The Novikov-Veselov Equation: Theory and Computation

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ABSTRACT. We review recent progress in theory and computation for the Novikov-Veselov (NV) equation with potentials decaying at infinity, focusing mainly on the zero-energy case. The inverse scattering method for the zero-energy NV equation is presented in the context of Manakov triples, treating initial data of conductivity type rigorously. Special closed-form solutions are presented, including multisolitons, ring solitons, and breathers. The computational inverse scattering method is used to study zero-energy exceptional points and the relationship between supercritical, critical, and subcritical potentials.

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1. Introduction

The Novikov-Veselov (NV) equation is the completely integrable, nonlinear dispersive equation

(1.1)
$$q_t = 4 \operatorname{Re} \left(4\partial^3 q + \partial \left(qw \right) - E \partial w \right)$$
$$\overline{\partial} w = -3\partial q$$

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Here E is a real parameter, the unknown function q is a real-valued function of two space variables and time, and the operators ∂ and $\overline{\partial}$ are given by

$$\partial = \frac{1}{2} \left(\partial_{x_1} - i \partial_{x_2} \right)$$
$$\overline{\partial} = \frac{1}{2} \left(\partial_{x_1} + i \partial_{x_2} \right).$$

At zero energy (E = 0) it can also be written (after trivial rescalings) as

(1.2)
$$q_t = -\partial_z^3 q - \overline{\partial}^3 q + 3\partial_z(q\nu) + 3\overline{\partial}_z(q\overline{\nu}), \quad \text{where } \overline{\partial}_z\nu = \partial_z q$$

The NV equation (1.2) generalizes the celebrated Korteweg-de Vries (KdV) equation

$$q_t = -6qq_x - q_{xxx}$$

in the sense that, if $q(x_1, t)$ solves KdV and $\nu_{x_1}(x_1, t) = -3q_{x_1}(x_1, t)$, then $q(x_1, t)$ solves NV.

The NV equation was introduced by Novikov and Veselov in [64, 65] as one of a hierarchy of completely integrable equations that generate isospectral flows for the two-dimensional Schrödinger operator at fixed energy E. Indeed, the Novikov-Veselov equation (1.1) admits the Manakov Triple Representation [48]

$$\dot{L} = [A, L] - BL$$

where

$$L = -\Delta + q - E,$$

$$A = 8 \left(\partial^3 + \overline{\partial}^3 \right) + 2 \left(w \partial + \overline{w} \overline{\partial} \right),$$

$$B = -2 \left(\partial w + \overline{\partial} \overline{w} \right).$$

Here B is the operator of multiplication by the function $2(\partial w + \overline{\partial w})$. That is, a pair (q, w) solves the NV equation if and only if the operator equation (1.3) holds.

The Manakov triple representation implies that the NV equation is, formally at least, a completely integrable equation. Thus one expects that, for a suitable notion of "scattering data for L at fixed energy E," the associated scattering transform will linearize the flow.

For nonzero energy E and potentials q which vanish at infinity, the scattering transform and inverse scattering method was developed by P. Grinevich, R. G. Novikov, and S.-P. Novikov (see Kazeykina's thesis [36] for an excellent survey and see [23, 26–28] for the original papers). Roughly and informally, there is a scattering transform \mathcal{T} which maps the potential q to scattering data that obey a linear equation if q obeys the NV equation, and an inverse scattering transform \mathcal{Q} which inverts \mathcal{T} , so that the function

(1.4)
$$q(x,\tau) = \mathcal{Q}\left[e^{i\left(\diamond^3 + \left(\bar{\diamond}\right)^3\right)\tau}\mathcal{T}(q_0)\right](x)$$

solves the NV equation with initial data q_0 . The inverse scattering method may be visualized by the following commutative diagram:

where \mathcal{T} and \mathcal{Q} stand for the direct and inverse nonlinear Fourier transform, respectively, and the function $\mathbf{t}_{\tau} : \mathbb{C} \to \mathbb{C}$ is called the *scattering transform*. In the case E = 0, the inverse scattering method was studied by Boiti et. al. [9], Tsai [89], Nachman [58], Lassas-Mueller-Siltanen [44], Lassas-Mueller-Siltanen-Stahel [45, 46], Music [54], Music-Perry [55], and Perry [68]. Recently, Angelopoulos [1] proved that the Novikov-Veselov equation at E = 0 is locally well-posed in the Sobolev spaces $H^s(\mathbb{R}^2)$ for s > 1/2, placing the local existence theory for this equation on a sound footing. The potential utility of the inverse scattering method is to elucidate global behavior of the solutions.

The case E = 0 is somewhat special and is intimately connected with the following trichotomy of behaviors for the two-dimensional Schrödinger operator L at zero energy.

DEFINITION 1.1. The operator $L = -\Delta + q$ is said to be: (i) subcritical if the operator L has a positive Green's function and the equation

 $L\psi = 0$ has a strictly positive distributional solution,

(ii) critical if $L\psi = 0$ has a bounded strictly positive solution but no positive Green's function, and

(iii) supercritical otherwise.

This distinction first arose in the study of Schrödinger semigroups, i.e., the operators e^{-tL} where $L = -\Delta + q$. Simon [**79**, **80**] (see also [**81**]) studied L^{p} -mapping properties of e^{-tL} and asymptotics of $||e^{-tL}||_{p,p}$, where $|| \cdot ||_{p,p}$ denotes the operator norm as maps from L^{p} to itself. Simon shows that

$$\alpha_p(q) = \lim_{t \to \infty} t^{-1} \ln \left\| e^{-tL} \right\|_{p,p}$$

is independent of $p \in [1, \infty]$. In the language of Schrödinger semigroups, a potential q is:

(i) subcritical if $\alpha_{\infty} ((1 + \varepsilon) q) = 0$ for some $\varepsilon > 0$,

(ii) critical if $\alpha_{\infty}(q) = 0$ but $\alpha_{\infty}((1 + \varepsilon)q) > 0$ for all $\varepsilon > 0$, and

(iii) supercritical if $\alpha_{\infty}(q) > 0$.

Clearly, a sufficient condition for q to be supercritical is that L have a negative eigenvalue.

In [52], Murata showed that, for two-dimensional Schrödinger operators with potentials q with q(x) uniformly Hölder continuous and $q(x) = \mathcal{O}(|x|^{-4-\epsilon})$ for some $\epsilon > 0$, the trichotomy of behaviors for Schrödinger semigroups is equivalent to Definition 1.1. Murata further studied the existence and properties of positive solutions of the Schrödinger equation in [52], and showed that for his class of potentials, the trichotomy could be characterized as follows: a potential is

(i) subcritical if and only if $L\psi = 0$ has a strictly positive solution of the form $c \log(|x|) + d + \mathcal{O}(|x|^{-1})$ with $c \neq 0$,

(ii) critical if $L\psi = 0$ if and only if $L\psi = 0$ has a strictly positive bounded solution, and

(iii) supercritical if $L\psi = 0$ has no strictly positive solutions.

Extending Murata's result, Gesztesy and Zhao [20] used Brownian motion techniques to prove the following optimal result for critical potentials. Suppose that q is a real-valued measurable function with

$$\lim_{\alpha \downarrow 0} \left\{ \sup_{x \in \mathbb{R}^2} \int_{|x-y| \le \alpha} \ln\left(|x-y|\right)^{-1} \left| q(y) \right| \, dy \right\} = 0$$

and

 $\int_{|y|\ge 1} \ln(|y|) |q(y)| \, dy < \infty.$

Then q is critical if and only if there exists a positive, bounded distributional solution ψ of $H\psi = 0$. These two conditions mean essentially that

$$q(x) = \mathcal{O}\left(|x|^{-2} \left(\ln(|x|)\right)^{-2-\epsilon}\right)$$

for some $\epsilon > 0$. We refer the reader to [20] for further references and history.

As we will see, corresponding to the trichotomy in Definition 1.1, the scattering transform of q is either mildly singular, nonsingular, or highly singular. This is illustrated dramatically in the examples studied by Music, Perry, and Siltanen [56], described in Section 5 below. One would expect the singularities of the scattering transform to be mirrored in the behavior of solutions to the NV equation. We will discuss the following conjecture, and some partial results toward its resolution, in the last section of this paper:

CONJECTURE 1.2. The Novikov-Veselov equation (1.1) has a global solution for critical and subcritical initial data, but its solution may blow up in finite time for supercritical initial data.

To elucidate this conjecture, it is helpful to recall how the scattering transform for Schrödinger's equation is connected with Calderón's inverse conductivity problem [15]. Critical potentials are also known in the literature as "potentials of conductivity type" because of their connection with the Calderón inverse conductivity problem: suppose one wishes to determine the conductivity σ of a bounded plane region Ω by boundary measurements. The potential u of Ω with voltage fon the boundary is determined by the equation

$$abla \cdot (\sigma \nabla u) = 0$$

 $u|_{\partial \Omega} = f$

Calderón's problem is to reconstruct σ from knowledge of the *Dirichlet-to-Neumann* map, defined as follows. If Ω has smooth boundary then the above boundary value problem has a unique solution u for given $f \in H^{1/2}(\partial\Omega)$, so that

$$\Lambda_{\sigma}f = \sigma \left. \frac{\partial u}{\partial \nu} \right|_{\partial \Omega}$$

is uniquely determined. The map $\Lambda_{\sigma}: H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is the Dirichlet-to-Neumann map.

This boundary value problem is equivalent, under the change of variables $u = \sigma^{-1/2}\psi$, to the Schrödinger problem

$$\Delta \psi - q\psi = 0$$

where $q = \sigma^{-1/2} \Delta(\sigma^{1/2})$. A potential of this form for strictly positive $\sigma \in L^{\infty}$ (and some additional regularity) is called a *potential of conductivity type*. More precisely, the class of potentials originally studied by Nachman [58] is as follows. We denote by $L^p_{\rho}(\mathbb{R}^2)$ the space of measurable functions with norm $\|f\|_{L^p_{\rho}} = \|\langle x \rangle^{\rho} f\|_p$.

DEFINITION 1.3. Let $p \in (1,2)$ and $\rho > 1$. A real-valued measurable function $q \in L^p_{\rho}(\mathbb{R}^2)$ is a potential of conductivity type if there is a function $\sigma \in L^{\infty}(\mathbb{R}^2)$ with $\sigma(x) \geq c_0 > 0$ so that $q = (\Delta(\sigma^{1/2}))/(\sigma^{1/2})$ in the sense of distribution derivatives.

A real-valued potential in $L^p_{\rho}(\mathbb{R}^2)$ is of conductivity type if and only if it is critical: the bounded, positive solution to $\Delta \psi - q\psi = 0$ is exactly $\psi = \sigma^{1/2}$.

As shown by Murata [52,53], critical potentials are very unstable: if $w \in C_0^{\infty}(\mathbb{C})$ is a nonnegative bump function and q_0 is a critical potential, the potential $q_{\lambda} = q_0 + \lambda w$ is supercritical for any $\lambda < 0$. This means that the set of critical potentials is nowhere dense in any reasonable function space! Music, Siltanen, and Perry [56] studied the behavior of the scattering transform for families of this type when q_0 and w are both smooth, compactly supported, and radial. The corresponding scattering transforms are mildly singular for subcritical potentials, regular for critical potentials, and have circles of singularities for supercritical potentials.

The NV equation may be solved by inverse scattering for subcritical and critical potentials, but it is not yet clear how to construct a solution by inverse scattering for supercritical potentials.

In this article, we will focus primarily on the Novikov-Veselov equation at zero energy. We will report on recent progress on both the theoretical and the numerical analysis of this equation, and pose a number of open problems. In section 2 we review the history of the inverse scattering method, the dispersion relation, symmetries and scaling properties, and conservation laws for the NV equation. In section 3, we give an exposition of the inverse scattering method for the NV equation at zero energy from the point of view of the Manakov triple representation, treating with full mathematical rigor the case of "smooth potentials of conductivity type" (see Definition 3.2). We discuss the numerical implementation of the maps \mathcal{T} and \mathcal{Q} in sections 3.2 and 3.3, respectively. In section 4, we discuss special closed-form solutions of the NV equation including ring solitons and breathers. In section 5, the computational inverse scattering method is used to study zero-energy exceptional points and the relationship between supercritical, critical, and subcritical potentials. Finally, in section 6, we discuss open problems. In an appendix, we collect some useful tools for the mathematical analysis of the direct and inverse scattering maps.

Notation. In what follows, we use the variable t to denote time except when discussing the solution of NV via the inverse scattering method. In this case, τ denotes time in order to distinguish t from \mathbf{t} , the scattering transform.

2. Background for the Zero-Energy NV Equation

First, we summarize the historical progress that led to the completion of the diagram (1.5) for the NV equation at zero energy. In 1987, Boiti, Leon, Manna and Pempinelli [9] studied the evolution under the assumption that q_0 is such that the solution $q_{\tau}^{\rm NV}$ to (1.1) exists and does not have exceptional points and established that the scattering data evolves as

$$\mathcal{T}(q_{\tau}^{\mathrm{NV}}) = e^{i\tau(k^3 + \overline{k}^3)} \mathcal{T}(q_0).$$

In 1994, Tsai [89] considered a certain class of small and rapidly decaying initial data (which excludes conductivity-type potentials) and assumed that q_0 has no exceptional points and that q_{τ} is well-defined. Under such assumptions, he then showed that q_{τ} is a solution of the Novikov-Veselov equation (1.1). In 1996, Nachman [58] established that initial data of conductivity type does not have exceptional points and the scattering data $\mathcal{T}(q_0)$ is well-defined. Nachman's work paved the way for rigorous results: all studies about diagram (1.5) published before [58] were formal as they had to assume the absence of exceptional points without specifying acceptable initial data. In 2007, Lassas, Mueller and Siltanen [44] established for smooth, compactly supported conductivity-type initial data with $\sigma \equiv 1$ outside $\operatorname{supp}(q_0)$ that there is a well-defined continuous function $q_\tau: \mathbb{R}^2 \to \mathbb{C}$ from the inverse scattering method satisfying the estimate $|q_{\tau}(z)| \leq C(1+|z|)^{-2}$ for all $\tau > 0$. In [45] it was shown that an initially radially-symmetric conductivity-type potential evolved under the ISM does not have exceptional points and is itself of conductivity-type. Note that in [56] the set of conductivity type potentials is shown to be unstable under C_0^{∞} perturbations. In [46] evolutions computed from a numerical implementation of the inverse scattering method of rotationally symmetric, compactly supported conductivity type initial data are compared to evolutions of the NV equation computed from a semi-implicit finite-difference discretization of NV and are found to agree with high precision. This supported the integrability conjecture that was then established in [67] for a larger class of initial data using the inverse scattering map for the Davey-Stewartson equation and Bogdanov's Miura transform.

In Section 4 of this paper, we present several closed-form solutions of the NV equation. We briefly review earlier constructions of solutions for the NV equation without presenting an exhaustive list. Grinevich, Manakov and R. G. Novikov constructed solition solutions using nonlocal Riemann problem techniques in [22-24,**24,27** for nonzero energy and with small initial data. Also, solitons are constructed by Grinevich using rational potentials in [22], by Tagami using the Hirota method in [82], by Athorne and Nimmo using Moutard transformation in [2], by Hu and Willox using a nonlinear superposition formula in [32], by Xia, Li and Zhang using hyperbola function method and Wu-elimination method in [95], by Ruan and Chen using separation of variables in [71–73,98], and by J.-L. Zhang, Wang, Wang and Fang using the homogeoneous balance principle and Bäcklund transformation in [32]. Lump solutions are constructed by Dubrovsky and Formusatik using the ∂ -dressing method in [16]. Dromion solutions are constructed by Ohta and Unal using Pfaffians in [66, 90]. The Darboux transformation is used by Hu, Lou, Liu, Rogers, Konopelchenko, Stallybrass and Schief to construct solutions in [33, 70]. Taimanov and Tsarëv [84–88] use the Moutard transformation to construct examples of Schrödinger operators L with L^2 eigenvalues at zero energy, and solutions of NV which blow up in finite time. In [96,97] Zheng, J.-F. Zhang and Sheng explore chaotic and fractal properties of solutions to the NV equation.

2.1. Dispersion, group velocity and phase velocity. Solitons form when there is a balance between nonlinearity and dispersion. The dispersion relation is the relation that gives the frequency as a function of the wave vector (k_1, k_2) . To find the dispersion relation for the NV equation, consider the linear part of the equation

(2.1)
$$q_t = -\frac{1}{4}q_{xxx} + \frac{3}{4}q_{xyy}$$

The plane wave functions $q(x, y, t) = e^{i(k_1x+k_2y-\omega t)}$ are solutions to (2.1) provided

(2.2)
$$\omega = -\frac{1}{4}k_1^3 + \frac{3}{4}k_1k_2^2.$$

Equation (2.2) defines the dispersion relation for the NV equation.



FIGURE 1. Surface plot (left) and heat map (right) of the NV dispersion relation $\omega(k) = k_1^2/4 + 3k_1k_2^2/4$

The phase velocity, \mathbf{c}_p , which gives the velocity of the wavefronts, is defined by $\mathbf{c}_p = \frac{\omega(\mathbf{k})}{|\mathbf{k}|^2} (k_1, k_2)^T$ and for the NV equation is

(2.3)
$$\mathbf{c}_p = \frac{k_1^3 - 3k_1k_2^2}{4(k_1^2 + k_2^2)} (k_1, k_2)^T.$$

The group velocity, which gives the velocity of the wave packet, is

$$\mathbf{c}_g \equiv \nabla \omega = \frac{3}{4} \left(-k_1^2 + k_2^2, \, 2k_1 k_2 \right)^T$$

2.2. Symmetries and Scaling. To understand how scaling of the dependent and independent variables change the NV equation, let us first consider the auxilliary $\overline{\partial}$ equation in equation (1.2) in the form

$$\partial \nu = \partial q, \quad \nu = v + iw.$$

Note that under the transformation $r(x, y, t) \equiv \gamma \nu(\alpha t, \beta x, \beta y)$ and $s(x, y, t) \equiv \gamma q(\alpha t, \beta x, \beta y)$, the $\overline{\partial}$ equation remains unchanged, i.e $\overline{\partial} \nu = \partial q$ if and only if $\overline{\partial} r = \partial s$. Then $r_x(x, y, t) = \beta \gamma \nu_x(\alpha t, \beta x, \beta y)$ and $r_y(x, y, t) = \beta \gamma \nu_y(\alpha t, \beta x, \beta y)$.

Now, we examine the main equation as presented in equation (1.2). Note that $s_t(x, y, t) = \alpha \gamma q_t(\alpha t, \beta x, \beta y), \ s_{xxx}(x, y, t) = \beta^3 \gamma q_{xxx}(\alpha t, \beta x, \beta y), \ s_{xyy}(x, y, t) =$

 $\beta^3 \gamma q_{xyy}(\alpha t, \beta x, \beta y)$, and $(qv)_x + (qw)_y = \gamma^2 \beta((s\Re(r))_x + (s\Im(r)_y))$. Assuming q is a solution to the NV equation (1.1), we find

$$4q_t = -u_{xxx} + 3q_{xyy} + 3(qv)_x + 3(qw)_y$$
$$\implies \frac{4}{\alpha\gamma}s_t = -\frac{1}{\beta^3\gamma}s_{xxx} + \frac{3}{\beta^3\gamma}s_{xyy} + \frac{3}{\beta\gamma^2}(s\Re(r))_x + \frac{3}{\beta\gamma^2}(s\Im(r))_y.$$

Multiplying by $\alpha \gamma$ leads to

(2.4)
$$4s_t = -\frac{\alpha}{\beta^3}s_{xxx} + \frac{3\alpha}{\beta^3}s_{xyy} + \frac{3\alpha}{\beta\gamma}((sr_1)_x + (sr_2)_y).$$

The table below shows the possible sign conventions possible for each term of the right hand side of equation (2.4).

| α | β | γ | Signs in (2.4) | | |
|----------|---------|----------|------------------|---|---|
| + | + | + | - | + | + |
| - | - | + | - | + | + |
| - | + | + | + | - | - |
| + | - | + | + | - | - |
| + | + | - | - | + | - |
| - | - | - | - | + | - |
| - | + | - | + | - | + |
| + | - | - | + | - | + |

Thus, there is a fixed ratio of -3 of the coefficients of the linear spatial terms, and any other coefficient is possible by proper rescaling of independent and dependent variables.

We also consider under what rotations the Novikov-Veselov equation is invariant. Writing the NV equation as

(2.5)
$$q_t = -\partial^3 q - \overline{\partial^3} q + 3\partial(q\nu) + 3\overline{\partial}(q\overline{\nu}),$$

(2.6)
$$\overline{\partial}\nu = \partial q$$

it is easy to see by conjugating (2.5), that if q is real at time t_0 , then q stays real.

If the initial value $q_0(z)$ is invariant under rotations by $\pm \frac{2\pi}{3}$, then the Novikov-Veselov evolution preserves this symmetry, see [46]. In particular, all radially symmetric initial values will lead to solutions with this three-fold symmetry.

Under such rotations, equation (2.5) becomes

(2.7)
$$q_t = e^{i\theta}\partial_{z'}(\nu'q) + e^{i\theta}\overline{\partial}_{z'}(\overline{\nu'}q) - e^{3i\theta}\overline{\partial}_{z'}^3q - e^{-3i\theta}\partial_{z'}^3q$$

where $\nu' = e^{-i\theta}\nu$. The auxiliary equation becomes

$$e^{i\theta}\overline{\partial}_{z'}\nu = e^{-i\theta}\partial_{z'}q$$

or

$$\overline{\partial}_{z'}\nu' = e^{-3i\theta}\partial_{z'}q,$$

and so we have invariant solutions under rotations of $2\pi/3$ and $4\pi/3$. This shows that if a solution to the NV equation has this symmetry, it must be preserved under the evolution. It does not mean that all solutions will display this type of symmetry.

2.3. Conservation Laws for the NV equation. In order to present the conservation laws for (1.1), we need to recall some ideas from the inverse scattering method. A rigorous derivation of the conservation laws for smooth potentials of conductivity type is given below in section 3.

Suppose that $q \in L^p(\mathbb{R}^2)$ for some $p \in (1, 2)$. The scattering data, or scattering transform $\mathbf{t} : \mathbb{C} \to \mathbb{C}$ of q is defined via Faddeev's [18] complex geometric optics (CGO) solutions, which we now recall. Let z = x + iy and $k = k_1 + ik_2$. For $k \in \mathbb{C}$ with $k \neq 0$, the function $\psi(z, k)$ is the exponentially growing solution of the Schrödinger equation

(2.8)
$$(-\Delta + q)\psi(\cdot, k) = 0$$

with asymptotic behavior $\psi(z,k) \sim e^{ikz}$ in the following sense: for \tilde{p} defined by $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}$, we have

(2.9)
$$e^{-ikz}\psi(z,k) - 1 \in L^{\tilde{p}}(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$$

It is more convenient to work with the normalized complex geometric optics solutions (NCGO) $\mu(z, k)$ defined by

(2.10)
$$\mu(z,k) = \psi(z,k)e^{-ikz}$$

A straightforward computation shows that μ obeys the equation

(2.11)
$$\overline{\partial} (\partial + ik) \mu = \frac{1}{4} q \mu, \qquad \mu(\cdot, k) - 1 \in L^{\tilde{p}} \cap L^{\infty}.$$

One can reduce the problem (2.11) to an integral equation of Fredholm type (see the discussion in section 3). Faddeev's Green's function g_k is the fundamental solution for the equation

$$-4\overline{\partial}(\partial + ik)u = f$$

(the factor of -4 is chosen so that, if k = 0, the equation reduces to $-\Delta u = f$ whose fundamental solution is the logarithmic potential; see Appendix A.1 for details). One has

(2.12)
$$\mu = 1 - g_k * (q\mu)$$

(where * denotes convolution) and it can be shown that the operator $\varphi \mapsto g_k * (q\varphi)$ is compact on $L^{\tilde{p}}$. Thus, for given k, the solution $\mu(z,k)$ exists if and only if it is unique.

It is known that such solutions exist for any $q \in L^p$ provided that |k| is sufficiently large. In general, however, the equation (2.11) need not have a unique solution for every k.

Points for which uniqueness fails, i.e., points for which the homogeneous problem

$$\begin{split} \overline{\partial} \left(\partial + ik \right) \mu &= \frac{1}{4} q \mu, \\ \mu(\cdot,k) \in L^{\tilde{p}} \cap L^{\infty} \end{split}$$

has a nontrivial solution are called *exceptional points*. It is known that the exceptional points form a bounded closed set in \mathbb{C} . Nachman [58] proved that the exceptional set is empty for potentials of conductivity type; more recently, Music [54] has shown that the same is true for subcritical potentials. We discuss this further in section 5.

Let

(2.13)
$$e_k(z) = e^{i\left(kz+k\overline{z}\right)}.$$

Then, if q decays rapidly at infinity (say $q \in \mathcal{S}(\mathbb{R}^n)$), the function $\mu(z, k)$ obeys the large-z asymptotic formula

(2.14)
$$\mu(z,k) \sim 1 + \frac{1}{4\pi i k z} s(k) - \frac{e_{-k}(z)}{4\pi i \overline{k} \overline{z}} \mathbf{t}(k) + \mathcal{O}\left(|z|^{-2}\right)$$

where

(2.15)
$$\mathbf{t}(k) = \int e_k(z)q(z)\mu(z,k) \, dz,$$

(2.16)
$$s(k) = \int q(z)\mu(z,k) dz.$$

Note that s(k) and $\mathbf{t}(k)$ are *always* defined for large |k|, whether the potential q is subcritical, critical, or supercritical.

The asymptotic formula (2.14) is a consequence of the following simple lemma.

LEMMA 2.1. Suppose $k \in \mathbb{C}$ and $k \neq 0$, that p > 2 and suppose that $u \in C^2(\mathbb{C}) \cap L^p(\mathbb{C})$ satisfies

$$-4\overline{\partial}\left(\partial+ik\right)u=f$$

for $f \in \mathcal{S}(\mathbb{C})$. Then

(2.17)
$$u(z) \sim_{|z| \to \infty} \sum_{\ell \ge 0} \frac{a_{\ell}}{z^{\ell+1}} + e_{-k}(z) \frac{b_{\ell}}{z^{\ell+1}}$$

where

$$a_0 = \frac{1}{4\pi i k} \int f(z) \, dz,$$

$$b_0 = \frac{1}{4\pi i \overline{k}} \int e_k(z) f(z) \, dz.$$

PROOF. The conditions on u imply that

$$u = g_k * f$$

so using the asymptotic expansion (A.3) for g_k we obtain (2.18)

$$u(z) = -\frac{1}{4\pi} \sum_{j=0}^{N} \int \left[\frac{j!}{(ik(z-z'))^{j+1}} + e^{-i(kz+\overline{k}\overline{z})} \frac{j!}{(-i\overline{k}(\overline{z-z'}))^{j+1}} \right] f(z') dz' + \mathcal{O}\left(|z|^{-N-2}\right)$$

It is not difficult to see that for $f \in \mathcal{S}(\mathbb{C})$ and any positive integer N, the expansion

$$\int \frac{1}{(z-z')^{j}} f(z') \, dz' = \sum_{\ell=0}^{N} c_{\ell,j} z^{-j-\ell} + \mathcal{O}\left(|z|^{-N-1-j}\right)$$

holds, with an analogous expansion for the terms involving $\overline{z} - \overline{z'}$. The existence of the expansion (2.17) is immediate. The leading terms come from the j = 0 term of (2.18).

If $q(z, \tau)$ solves the NV equation at zero energy, it can be shown that $\mathbf{t}(\cdot, \tau)$, the scattering transform of $q(\cdot, \tau)$, is then given by

(2.19)
$$\mathbf{t}(k,\tau) = \mathbf{m}(k,\tau)\mathbf{t}(k,0),$$

where

$$\mathbf{m}(k,\tau) = \exp(i\tau(k^3 + \overline{k}^3)).$$

On the other hand,

(2.20)
$$s(k,\tau) = s(k,0).$$

Here $\mathbf{t}(k,\tau)$ and $s(k,\tau)$ are computed from the solution $\mu(z,k,\tau)$ of

(2.21)
$$\overline{\partial}_z \left(\partial_z + ik\right) \mu(z,k,\tau) = \frac{1}{4} q(z,\tau) \mu(z,k,\tau).$$

A rigorous proof of (2.19) and (2.20) for smooth potentials of conductivity type is given in section 3 below.

If q is smooth, rapidly decreasing, and either critical or subcritical, the Schrödinger potential q can be recovered using the $\overline{\partial}$ -method of Beals and Coifman [5] (see [44] for the critical case, and [54] for the subcritical case). Both of these papers use techniques developed by Nachman [58] in the context of the inverse conductivity problem.

We can now derive a set of conservation laws for the NV equation by using the large-k asymptotic expansion of s(k). Since $s(k,\tau)$ is conserved we set $\tau = 0$ and suppress τ -dependence henceforward. We will give a formal derivation of the conserved quantities by using a large-k asymptotic expansion of $\mu(z, k)$ and inserting this expansion into the formula (2.16). Since s(k) is conserved, the coefficients of that large-k expansion are also conserved quantities. For the moment, we assume that $\mu(z, k)$ admits a large-k expansion of the form

(2.22)
$$\mu(z,k) \sim 1 + \sum_{j=1}^{\infty} \frac{a_j(z)}{k^j}.$$

We will derive such an expansion for smooth potentials of conductivity type in the next section (see Lemma 3.7). It is expected to hold in general.

Substituting the series (2.22) into (2.21), we may solve the resulting system for the coefficients a_j

(2.23)
$$-\sum_{j=1}^{\infty} \frac{\Delta a_j(z)}{k^j} - \sum_{j=1}^{\infty} \frac{4i\overline{\partial} a_j(z)}{k^{j-1}} + q \sum_{j=1}^{\infty} \frac{a_j(z)}{k^j} = -q.$$

We find

$$a_1 = \frac{1}{4i}\overline{\partial}^{-1}q$$

A recursion formula can then be derived,

(2.24)
$$a_{j+1} = \frac{1}{4i}\overline{\partial}^{-1}(-4\overline{\partial}\partial a_j + qa_j) = i\partial a_j + \frac{1}{4i}\overline{\partial}^{-1}(qa_j).$$

From this, it is clear that

$$s(k) \sim \sum_{j=0}^{\infty} \frac{s_j}{k^j}, \quad s_j = \int_{\mathbb{R}^2} q(z) a_j(z) \, dz.$$

Thus, the first three conserved quantities are

$$s_{0} = \int_{\mathbb{R}^{2}} q(z)dz,$$

$$s_{1} = \int_{\mathbb{R}^{2}} \frac{1}{4i}q(z)(\overline{\partial}^{-1}q)(z)dz,$$

$$s_{2} = \int_{\mathbb{R}^{2}} \left(\frac{1}{4}q(z)v(z) - \frac{1}{16}q\overline{\partial}^{-1}(q\overline{\partial}^{-1}q)(z)\right)dz,$$

where with z = x + iy and $\zeta = \zeta_1 + i\zeta_2$,

$$(\overline{\partial}^{-1}q)(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{q(\zeta)d\zeta}{z-\zeta}.$$

3. Inverse Scattering via Manakov Triples

In this section we develop the inverse scattering method to solve the Cauchy problem for the Novikov-Veselov (NV) equation at zero energy

(3.1)
$$q_{\tau} = 2 \operatorname{Re} \left(\partial^{3} q - \frac{3}{4} \partial(uq) \right)$$
$$\overline{\partial} u = \partial q$$
$$q|_{\tau=0} = q_{0}$$

for smooth Cauchy data q_0 of conductivity type (see Definition 3.2). Note that our convention for the NV equation differs slightly from (1.1); the form used here is more convenient for the zero-energy inverse scattering formalism. This section should be regarded as expository and the material here is undoubtedly "well known to the experts" (see the original paper of Manakov [48] and see e.g. Boiti, Leon, Manna, and Pempenelli [9] for the NV equation), although we give an essentially self-contained and mathematically rigorous presentation. An extension of these ideas to broader classes of potentials will appear in [54], [55], and [69].

In this section, we draw on previous work of Lassas, Mueller, and Siltanen [44], Lassas, Mueller, Siltanen, and Stahel [45], and Perry [67], particularly for mapping properties of the scattering transform and its inverse on the space of smooth functions of conductivity type as defined below. The main ingredient in our analysis (as contrasted to [44, 45, 67]) is the systematic use of the Manakov triple representation for the NV equation.

To describe the Manakov triple representation, suppose that q is a smooth function of z and $t\tau$. Suppose that there is a smooth function $u(z,\tau)$ with the property that $\overline{\partial}u = \partial q$ (existence of such a function for suitable classes of q can be deduced from properties of the Beurling transform; see the Appendix). Define¹

$$(3.2) L = \partial \partial - q/4,$$

(3.3)
$$A = \partial^3 + \overline{\partial}^3 - \frac{3}{4} \left(u\partial + \overline{u}\overline{\partial} \right),$$

(3.4)
$$B = \frac{3}{4} \left(\partial u + \overline{\partial} u \right),$$

 $^{^{1}}$ This Manakov triple differs from that of the introduction by numerical factors since we use, for convenience, the version (3.1) of the NV equation.

where B is a multiplication operator. The (L, A, B) representation means the following:

PROPOSITION 3.1. Let $q \in C^1([0,T]; C^{\infty}(\mathbb{C}))$ and suppose that there is a smooth function u with $\overline{\partial}u = \partial q$. Then q is a classical solution to the Cauchy problem (3.1) if and only if the operator identity

$$\dot{L} = [A, L] - BI$$

holds.

The proof is a straightforward calculation. The significance of the Manakov triple is that it defines a scattering problem at zero energy for the operator L, and a law of evolution of scattering data through the operator A. We will fully describe the inverse scattering method for smooth initial data of conductivity type, defined as follows:

DEFINITION 3.2. A function $q_0 \in \mathcal{C}^{\infty}(\mathbb{C})$ is a smooth function of conductivity type if

$$\frac{q_0}{2} = 2\partial u_0 + \left|u_0\right|^2$$

for some $u_0 \in \mathcal{S}(\mathbb{C})$ with $\partial u_0 = \partial \overline{u_0}$.

The regularity requirements can be considerably relaxed but we make them here to ease the exposition. To compare this definition with Nachman's definition (see Definition 1.3 in the introduction), one should think of u_0 as $2\overline{\partial} \log \sigma$.

We develop in turn the direct scattering transform, the inverse scattering transform, and the solution formula for NV. We also comment on numerical methods for implementing the direct and inverse scattering transforms.

3.1. The Direct Scattering Map. To compute the scattering transform of q_0 , one first constructs the complex geometric optics (CGO) solutions to (2.8). Analytically, it is more convenient to study the normalized complex geometric optics (NCGO) solutions μ defined by (2.10). As shown by Nachman [58], there exists a unique solution of (2.11) for every nonzero k, so that $\mathbf{t}(k)$ is defined for every nonzero k. Nachman also shows that $\mathbf{t}(k)$ is $\mathcal{O}(|k|^{\varepsilon})$ as $k \to 0$ for conductivity-type potentials.

DEFINITION 3.3. The map $\mathcal{T} : q \mapsto \mathbf{t}$ defined by the problem (2.11) and the representation formula (2.15) is called the direct scattering map.

We will use without proof the following result of Lassas, Mueller, Siltanen, Stahel [45] (see also [68] for a different proof)

LEMMA 3.4. Suppose that q is a smooth function of conductivity type, and let **t** be the scattering transform of q. Then $\mathbf{t}(k)/\overline{k} \in \mathcal{S}(\mathbb{C})$.

3.2. Computation of Scattering Transforms. We describe two approaches for the computation of $\mathbf{t} = \mathcal{T}q$ for a given compactly supported and continuous q. The *LS method* is most accurate for k away from zero, and the *DN method* is more effective for k near zero. Matlab codes for both approaches are available at the webpage of the book [51].

Without loss of generality we can assume that $\operatorname{supp}(q) \subset \Omega$ where $\Omega \subset \mathbb{R}^2$ is the open unit disc. The LS method is based on the definition

(3.6)
$$\mathbf{t}(k) = \int_{\mathbb{R}^2} e^{i\overline{k}\overline{z}}q(z)\psi(z,k) \, dz,$$

and the DN method uses integration by parts to transform (3.6) into

(3.7)
$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\overline{k}\overline{z}} (\Lambda_q - \Lambda_0) \psi(\cdot, k) \, dS(z),$$

where Λ_q is the *Dirichlet-to-Neumann map* defined below. A rigorous derivation of formulas (3.6) and (3.7) was given by Nachman in [58].

The LS method requires numerical evaluation of the *complex geometrical op*tics solutions $\psi(z, k)$. Numerically, it is better to solve the Lippmann-Schwinger equation for μ

(3.8)
$$\mu(z,k) = 1 - \int_{\Omega} g_k(z-w)q(w)\mu(w,k) \, dw.$$

A rigorous solvability analysis for equation (3.8) can be found in [56–58]. Here g_k is the fundamental solution satisfying $(-\Delta - 4ikd)g_k(z) = \delta(z)$. The origin of g_k is Faddeev's 1965 article [18]. Computationally, $g_1(z)$ can be evaluated using the Matlab expression "exp(-1i*z).*real(expint(-1i*z))/(2*pi);" The symmetry relation $g_k(z) = g_1(kz)$ extends this to all values $k \neq 0$. Note that g_k has a log k singularity when $k \to 0$, causing numerical difficulties for k near zero.

Equation (3.8) is defined in the whole plane $z \in \mathbb{R}^2$, so some kind of truncation is needed for practical computation. The first numerical computation of complex geometrical optics solutions was reported in [75] in the context of (3.8). That computation was used as a part of the first numerical implementation [76] of the $\overline{\partial}$ method for electrical impedance tomography. A more effective approach for computing μ is based on the periodization technique introduced by Gennadi Vainikko in [91]; see also [74, Section 10.5]. The adaptation of Vainikko's method to equation (3.8) was first introduced in [50]. For more details see [51, Section 14.3].

Now let us turn to the DN method. This method has practical use in the $\overline{\partial}$ reconstruction method for electrical impedance tomography. (See [51] and the relevant references therein.) We first define the Dirichlet-to-Neumann map Λ_q for the Dirichlet problem

(3.9)
$$(-\Delta + q) u = 0 \text{ in } \Omega$$
$$u|_{\partial\Omega} = f.$$

If zero is not a Dirichlet eigenvalue of $-\Delta + q$ in Ω , the problem (3.9) has a unique solution u for given $f \in H^{1/2}(S^1)$, and we set

(3.10)
$$\Lambda_q f = \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega}$$

where $\partial/\partial\nu$ denotes differentiation with respect to the outward normal on $\partial\Omega$.

Formula (3.7) requires the trace $\psi(\cdot, k)|_{\partial\Omega}$. According to [58], the traces can be solved from the boundary integral equation

(3.11)
$$\psi(z,k)|_{\partial\Omega} = e^{ikz} - \int_{\partial\Omega} G_k(z-w) \left(\Lambda_q - \Lambda_0\right) \psi(w,k)|_{\partial\Omega} \, dS(w),$$

if k is not an exceptional point of q. Here $G_k(z) := e^{ikz}g_k(z)$ is Faddeev's Green function for the Laplace operator. For details of the numerical solution of (3.11) see [51, Section 15.3].

3.3. The Inverse Scattering Map. It turns out the NCGO solutions $\mu(z, k)$ also solve a $\overline{\partial}$ -problem in the k variable determined by $\mathbf{t}(k)$. Letting

(3.12)
$$\mathbf{t}^{\sharp}(k) = \frac{\mathbf{t}(k)}{4\pi\overline{k}},$$

we have, for any p > 2:

LEMMA 3.5. Suppose that q is a smooth potential of conductivity type, and let $\mu(z,k)$ be the corresponding NCGO solutions. Then:

(3.13)
$$\overline{\partial}_{k}\mu(z,k) = e_{-k}\mathbf{t}^{\sharp}(k)\overline{\mu(z,k)}$$
$$\mu(z, \cdot) - 1 \in L^{p}(\mathbb{C}).$$

Moreover, q(z) is recovered from t and $\mu(z,k)$ by the formula

(3.14)
$$q(z) = \frac{4i}{\pi} \overline{\partial}_z \left(\int e_{-k}(z) \mathbf{t}^{\sharp}(k) \overline{\mu(z,k)} \, dk \right)$$

REMARK 3.6. This equation has at most one solution for each z, provided only that $\mathbf{t}^{\sharp} \in L^2$ by a standard uniqueness theorem for the $\overline{\partial}$ -problem (see Brown-Uhlmann [11], Corollary 3.11).

The fact that $\mu(z, k)$ obeys a $\overline{\partial}$ -equation in the k-variable also implies a large-k asymptotic expansion for $\mu(z, k)$.

LEMMA 3.7. Suppose that q is a smooth potential of conductivity type, and let $\mu(z,k)$ be the corresponding NCGO solution. Then

(3.15)
$$\mu(z,k) \sim 1 + \sum_{\ell \ge 0} \frac{c_{\ell}(z)}{k^{\ell+1}}$$

where

$$(3.16) c_0 = -\frac{i}{4}\overline{\partial}^{-1}q$$

(3.17)
$$c_1 = -\frac{1}{4}\partial\overline{\partial}^{-1}q + \frac{1}{16}\overline{\partial}^{-1}\left(q\overline{\partial}^{-1}q\right)$$

and the remaining c_j are determined by the recurrence

(3.18)
$$i\overline{\partial}c_{j+1} = \left(\frac{q}{4} - \partial\overline{\partial}\right)c_j$$

PROOF. The coefficients in the asymptotic expansion may be computed recursively from the equation $\overline{\partial} (\partial + ik) \mu = (q/4)\mu$ once the existence of the asymptotic expansion is established. To do so, note that

$$\mu(z,k) = 1 + \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{k-\kappa} e_{-k} \mathbf{t}^{\sharp}(\kappa) \overline{\mu(z,\kappa)} \, dm(\kappa).$$

Writing

$$(k-\kappa)^{-1} = k^{-1} \sum_{j=0}^{N} \left(\frac{\kappa}{k}\right)^{j} + \left(\frac{\kappa}{k}\right)^{N+1} \frac{1}{k-\kappa}$$

we obtain an expansion of the desired form with

$$c_j(z) = \frac{1}{\pi} \int \kappa^j e_{-k} \mathbf{t}^\sharp(\kappa) \overline{\mu(z,\kappa)} \, dm(\kappa)$$

and remainder

$$\frac{1}{\pi}k^{-(N+1)}\int \frac{1}{k-\kappa}\kappa^{N+1}e_k\mathbf{t}^{\sharp}(\kappa)\overline{\mu(z,\kappa)}\ dm(\kappa)$$

Finally to obtain explicit formulae for the c_j , we substitute the expansion (3.15) into (2.11) to obtain

$$i\overline{\partial}c_0 = \frac{1}{4}q$$

and the recurrence (3.18) from which (3.17) follows.

Motivated by these results, suppose given $\mathbf{t}^{\sharp} \in \mathcal{S}(\mathbb{C})$ and let $\mu(z,k)$ be the unique solution to (3.13). *Define* q by the reconstruction formula (3.14). Then the solution μ of (3.13) obeys the partial differential equation

$$\overline{\partial} \left(\partial + ik\right) \mu = \frac{q}{4}\mu, \qquad \lim_{|z| \to \infty} \mu(z, \cdot) - 1 = 0.$$

DEFINITION 3.8. The map $Q: \mathbf{t} \to q$ defined by (3.13) and (3.14) is called the inverse scattering map.

We conclude this subsection by obtaining a full asymptotic expansion for $\mu(z, k)$ which encodes relations between s and t.

LEMMA 3.9. Suppose that q is a smooth potential of conductivity type, and let \mathbf{t}^{\sharp} be given by (3.12). Then, the expansion

(3.19)
$$\mu(z,k) \underset{|z| \to \infty}{\sim} 1 + \sum_{\ell \ge 0} \left(\frac{a_\ell}{z^{\ell+1}} + e_{-k} \frac{b_\ell}{\overline{z^{\ell+1}}} \right)$$

holds, where:

$$a_{0} = -i\overline{\partial}_{k}^{-1}\left(\left|\mathbf{t}^{\sharp}\right|^{2}\right),$$
$$b_{0} = i\mathbf{t}^{\sharp}$$

and the subsequent a_{ℓ}, b_{ℓ} are determined by the recurrence relations

$$\begin{aligned} \overline{\partial}_k a_\ell &= \mathbf{t}^\sharp \overline{b_\ell}, \\ b_{\ell+1} &= i \overline{a_\ell} \mathbf{t}^\sharp - i \overline{\partial}_k b_\ell \end{aligned}$$

PROOF. The existence of an expansion of the form (3.19) was already established in Lemma 2.1. To compute the coefficients, we substitute the asymptotic series into (3.13).

REMARK 3.10. Comparing Lemmas 2.1 and 3.9, we see that

$$s(k) = 4\pi k \overline{\partial}_k^{-1} \left(\left| \mathbf{t}^{\#} \right|^2 \right)$$

3.4. Computation of Inverse Scattering Transforms. The first step in the computation of the inverse scattering map is to solve the $\overline{\partial}$ equation

(3.20)
$$\frac{\partial}{\partial \overline{k}} \mu_{\tau}(z,k) = \frac{\mathbf{t}_{\tau}(k)}{4\pi \overline{k}} e_{-k}(z) \overline{\mu_{\tau}(z,k)}$$

with a fixed parameter $z \in \mathbb{R}^2$ and requiring large |k| asymptotics $\mu_{\tau}(z, \cdot) - 1 \in L^{\infty} \cap L^r(\mathbb{C})$ for some $2 < r < \infty$. Since $q_0(z)$ is compactly supported and of conductivity type, by [44] the scattering transform $\mathbf{t}_{\tau}(k)$ is in the Schwartz class,

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and the solution μ_{τ} to equation (3.20) can be computed by numerically solving the integral equation

(3.21)
$$\mu_{\tau}(z,k) = 1 + \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{\mathbf{t}_{\tau}(k')}{\bar{k'}(k-k')} e_{-k'}(z) \overline{\mu_{\tau}(z,k')} dk'.$$

Computational solution of the $\overline{\partial}$ equation (3.20) is based on truncating the scattering transform $\mathbf{t}_{\tau}(k)$ to a large disc of radius R, generally chosen by inspection of the scattering transform. The truncated integral equation is solved numerically by the method described in [41] for each point z at which the evolved potential is to be computed. The method in [41] is based on Vainikko's periodic method [91]; see also [74, Section 10.5]. Note that since the d equation (3.20) is real-linear and not complex-linear due to the complex conjugate on the right-hand side of (3.20), one must write the real and imaginary parts of the unknown function μ separately in the vector of function values at the grid points. It is proven in [40] that the error decreases as R tends to infinity. The first computational solutions of equation (3.20) can be found in [76], and the first computations based on [91] are found in [41]; for more details see [51, Section 15.4].

The inverse scattering transform is defined by

(3.22)
$$(\mathcal{Q}\mathbf{t}_{\tau})(z) := \frac{i}{\pi^2} \overline{\partial}_z \int_{\mathbb{C}} \frac{\mathbf{t}_{\tau}(k)}{\overline{k}} e^{-ikz} \overline{\psi_{\tau}(z,k)} dk,$$

where $\psi_{\tau}(z,k) := e^{ikz} \mu_{\tau}(z,k)$. The inverse transform (3.22) first appeared in [9, formula (4.10)]. See [44] and the references therein for an analysis of the solvability of (3.20) and the domain of definition for (3.22). Under the assumption that real-valued, smooth initial data of conductivity type remain of conductivity type under evolution by the ISM, the conductivity γ_{τ} associated with the potential q_{τ} is given by

$$\gamma_{\tau}^{1/2}(z) = \mu_{\tau}(z,0).$$

Then q_{τ} is computed by numerical differentiation of γ_{τ} by the formula

$$q_{\tau}(z) = \gamma_{\tau}^{-1/2} \Delta \gamma_{\tau}^{1/2}$$

The reader is referred to [46] for numerical examples of the computation of the time evolution of conductivity-type potentials by the ISM.

3.5. Time Evolution of NCGO Solutions. In order to prove the solution formula (1.4), we first study the time evolution of NCGO solutions using the Manakov triple representation. First we note the following important uniqueness theorem which is actually a special case of results of Nachman.

THEOREM 3.11. Let $k \in \mathbb{C}$, $k \neq 0$. Suppose that q is a smooth potential of conductivity type and that ψ is a solution of $L\psi = 0$ with $\lim_{|z|\to\infty} (e^{-ikz}\psi(z)) = 0$. Then $\psi(z) = 0$.

Now suppose that $q(z, \tau)$ solves the NV equation and that $t \mapsto q(z, \tau)$ is a C^1 map from [0, T] into $\mathcal{S}(\mathbb{C})$. Suppose that, for each τ , $\varphi(\tau)$ solves $L(\tau)\varphi(\tau) = 0$. Differentiating the equation $L(\tau)\varphi(\tau) = 0$ and using the Manakov triple representation, we find

$$[A(\tau), L(\tau)]\varphi(\tau) + L(\tau)\dot{\varphi}(\tau) = 0$$

or

$$L(\tau) \left[\dot{\varphi}(\tau) - A(\tau)\varphi(\tau) \right] = 0$$

From this simple computation and Theorem 3.11, we can derive an equation of motion for the NCGO solutions, and recover (2.19) and (2.20) from a careful calculation of asymptotics. Later, we will show by explicit construction that, if q_0 is a smooth function of conductivity type, then there is a solution $q(z,\tau)$ of the NV equation so that $q(z,\tau)$ is smooth and of conductivity type for each τ .

LEMMA 3.12. Suppose that $q(z,\tau)$ is a solution of the NV equation where, for each τ , $q(z,\tau)$ is a smooth function of conductivity type. Let $u = \overline{\partial}^{-1} \partial q$. Then

(3.23)
$$\dot{\mu} = ik^{3}\mu + (\partial + ik)^{3}\mu + \overline{\partial}^{3}\mu - \frac{3}{4}u(\partial + ik)\mu - \frac{3}{4}\overline{u}\overline{\partial}\mu$$

PROOF. Before giving the proof we make several remarks. Since $\psi = e^{ikz}\mu$, the evolution equation (3.23) is equivalent to

(3.24)
$$\psi_{\tau} = ik^{3}\psi + \partial^{3}\psi + \overline{\partial}^{3}\psi - \frac{3}{4}\left(u\partial + \overline{u}\overline{\partial}\right)\psi.$$

Next, let $\varphi(z,k,\tau) = e^{iS}\mu(z,k,\tau)$ with $S(z,k,\tau) = kz - k^3t$. From the argument above we have

$$L(\tau) \left[\dot{\varphi}(\tau) - A(\tau)\varphi(\tau) \right] = 0$$

To conclude that $\dot{\varphi}(\tau) = A(\tau)\varphi(\tau)$, we must show that

$$\lim_{|z|\to\infty} \left(e^{-ikz} \left[\dot{\varphi}(\tau) - A(\tau)\varphi(\tau) \right] \right) = 0.$$

Write

 $f \sim_k g$

if

$$\lim_{|z|\to\infty} \left[e^{-ikz} \left(f - g \right) \right] = 0$$

Noting that $\mu - 1$ and its derivatives in z and \overline{z} vanish as $|z| \to \infty$, a simple calculation shows that

$$\dot{\varphi} - A\varphi \sim_k e^{iS} \left(-ik^3\mu - (\partial + ik)^3\mu \right) \\ \sim_k 0$$

Hence $\dot{\varphi} = A\varphi$ from which (3.24) follows.

Hence:

LEMMA 3.13. Suppose that $q(z,\tau)$ is a solution of the NV equation where, for each τ , $q(z,\tau)$ is a smooth function of conductivity type. Let $\mu(z,k,\tau)$ be the corresponding NCGO solution with

(3.25)
$$\mu(z,k,\tau) \sim 1 + \frac{1}{4\pi i k z} s(k,t) - \frac{e_{-k}(z)}{4\pi i \overline{k} \overline{z}} \mathbf{t}(k,t) + \mathcal{O}\left(|z|^{-2}\right).$$

Then

(3.26)
$$\dot{s}(k,t) = 0,$$

(3.27)
$$\mathbf{\dot{t}}(k,t) = i\left(k^3 + \overline{k}^3\right)\mathbf{t}(k,t).$$

PROOF. Substituting the asymptotic relation (3.25) into (3.23), we may compute, modulo terms of order z^{-2} ,

$$-\frac{1}{\pi i k z} \dot{\mathbf{t}} + \frac{e_{-k}(z)}{\pi i \overline{k} \overline{z}} \dot{s} = -\frac{e_{-k}(z)}{\pi i \overline{k} \overline{z}} \left(i k^3 + i \overline{k}^3\right) \mathbf{t}.$$

The computation uses the following facts. If

$$\mu(z,k,\tau) = 1 + \frac{a_0}{z} + e_{-k}\frac{b_0}{\overline{z}} + \mathcal{O}\left(|z|^{-2}\right)$$

then ("~" means "is asymptotic as $|z| \to \infty$ to" modulo $\mathcal{O}(|z|^{-2})$)

$$\begin{split} \partial^{3}\mu \sim e_{-k} \left(-ik\right)^{3} \frac{b_{0}}{\overline{z}}, \\ \overline{\partial}^{3}\mu \sim e_{-k} \left(-i\overline{k}\right)^{3} \frac{b_{0}}{\overline{z}}, \\ 3ik\partial^{2}\mu - 3k^{2}\partial\mu \sim 0 \end{split}$$

together with the fact that u defined by $\overline{\partial}u = \partial q$ satisfies $u = \mathcal{O}(|z|^{-2})$. The identities (3.26) and (3.27) are immediate.

3.6. Solution by Inverse Scattering. Motivated by the computations of the preceding subsection, we now consider the problem

(3.28)
$$\overline{\partial}_k \mu = e^{i\tau S} \mathbf{t}^\sharp \mu$$
$$\mu(z, \cdot, t) - 1 \in L^p(\mathbb{C})$$

for a function $\mu(z, k, \tau)$ and the putative reconstruction

(3.29)
$$q(z,\tau) = \frac{4i}{\pi} \overline{\partial}_z \left(\int_{\mathbb{C}} e^{i\tau S} \mathbf{t}^{\sharp}(k) \overline{\mu(z,k,\tau)} \, dk \right).$$

Here

(3.30)
$$S(z,k,\tau) = -\frac{1}{\tau} \left(kz + \overline{k}\overline{z} \right) + \left(k^3 + \overline{k}^3 \right)$$

and \mathbf{t}^{\sharp} is obtained from the Cauchy data q_0 . We will show that $q(z, \tau)$ solves the NV equation by deriving an equation of motion for $\mu(z, k, \tau)$ and using this equation to compute q_{τ} if q is given by (3.29) and μ is the unique solution of (3.28)

First, we establish an equation of motion for the solution μ of (3.28). Although this equation is the same equation as (3.23) for the solution of the direct problem, our starting point here is (3.28).

LEMMA 3.14. Suppose that $\mathbf{t}^{\sharp} \in \mathcal{S}(\mathbb{C})$ and μ solves (3.28). For each τ , define $q(z,\tau)$ by (3.29) and define $u(z,\tau)$ by $u = \overline{\partial}^{-1} \partial q$. Then

$$\dot{\mu} = ik^{3}\mu + (\partial + ik)^{3}\mu + \overline{\partial}^{3}\mu - \frac{3}{4}u(\partial + ik)\mu - \frac{3}{4}\overline{u}\overline{\partial}\mu$$

where ∂ and $\overline{\partial}$ denote differentiation with respect to the z and \overline{z} variables.

PROOF. Let

$$w = \dot{\mu} - \left(ik^{3}\mu + (\partial + ik)^{3}\mu + \overline{\partial}^{3}\mu - \frac{3}{4}u\left(\partial + ik\right)\mu - \frac{3}{4}\overline{u}\overline{\partial}\mu\right).$$

We will show that w = 0 in two steps. First, we show that

$$\overline{\partial}_k w = e^{i\tau S} \mathbf{t}^\sharp \overline{w}$$

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This is an easy consequence of the formulas

$$\overline{\partial}_{k} \left(\partial + ik \right) \mu = e^{itS} \mathbf{t}^{\sharp} \overline{\partial} \mu,$$
$$\overline{\partial}_{k} \left(\overline{\partial} \mu \right) = e^{i\tau S} \mathbf{t}^{\sharp} \overline{(\partial + ik)} \mu$$

and holds for any smooth function u.

Next, we show that for any fixed values of the parameters τ and z,

$$\lim_{|k|\to\infty}w(z,k,\tau)=0$$

Here we must choose $u = \overline{\partial}^{-1} \partial q$ in order for the assertion to be correct. Owing to Lemma 3.7 and the formula

$$w = \dot{\mu} - \left(\partial^{3}\mu + \overline{\partial}^{3}\mu + 3ik\partial^{2}\mu - 3k^{2}\partial\mu - \frac{3}{4}u\left(\partial + ik\right)\mu - \frac{3}{4}\overline{u}\overline{\partial}\mu\right),$$

we have

$$w = -3ik\partial^{2}\mu + 3k^{2}\partial\mu + \frac{3}{4}u\left(\partial + ik\right)\mu + \mathcal{O}\left(k^{-1}\right)$$
$$= A_{-1}k + A_{0} + \mathcal{O}\left(k^{-1}\right)$$

where

$$A_{-1} = 3\left(\partial a_0 + \frac{i}{4}u\right)$$

and

$$A_0 = 3\left[-i\partial^2 a_0 + \partial a_1 + \frac{i}{4}ua_0\right].$$

The condition $A_{-1} = 0$ forces the choice $u = \overline{\partial}^{-1} \partial q$. We may then compute

$$A_0 = \frac{3}{16} \left[\partial \overline{\partial}^{-1} \left(q \overline{\partial}^{-1} q \right) - \left(\overline{\partial}^{-1} q \right) \cdot \left(\partial \overline{\partial}^{-1} q \right) \right].$$

One the one hand, A_0 vanishes as $|z| \to \infty$ for each fixed τ by the decay of q. On the other hand, a straightforward computation shows that $\overline{\partial}A_0 = 0$. It now follows from Liouville's Theorem that $A_0 = 0$, and hence $w = O(k^{-1})$. We now used the generalized Liouville Theorem to conclude that w = 0.

Finally, we prove:

PROPOSITION 3.15. Suppose that $\mathbf{t}^{\sharp} \in \mathcal{S}(\mathbb{R}^2)$. Then, the formula

$$q(z,\tau) = \frac{4i}{\pi} \overline{\partial}_z \left(\int e^{itS(z,k,\tau)} \mathbf{t}^{\sharp}(k) \overline{\mu(z,k,\tau)} \ dm(k) \right)$$

yields a classical solution of the NV equation.

PROOF. In what follows, we will freely use the commutation relations

- (3.31) $\partial e^{i\tau S} = e^{i\tau S} \left(\partial ik\right),$
- (3.32) $\overline{\partial}e^{i\tau S} = e^{i\tau S} \left(\overline{\partial} i\overline{k}\right)$

and the equation

(3.33)
$$\overline{\partial} \left(\partial + ik \right) \mu = \frac{1}{4} q \mu$$

For notational brevity we'll write $c = 4i/\pi$. We compute

$$\begin{split} \dot{q} &= c\overline{\partial} \left(\int e^{i\tau S} \mathbf{t}^{\sharp} \left\{ ik^{3} + i\overline{k}^{3} \right\} \mu \right) \\ &+ c\overline{\partial} \left(\int e^{i\tau S} \mathbf{t}^{\sharp} \left\{ -i\overline{k}^{3} + \left(\overline{\partial} - i\overline{k}\right)^{3} + \partial^{3} - \frac{3}{4}\overline{u} \left(\overline{\partial} - i\overline{k}\right) - \frac{3}{4}u\partial \right\} \overline{\mu} \right) \end{split}$$

where in the second term we used Lemma 3.14. Using the commutation relations above to move differential operators to the left of $\exp(itS)$, we conclude that

$$\dot{q} = \partial^3 q + \overline{\partial}^3 q - \frac{3}{4} \overline{\partial} \left(u \partial \overline{\partial}^{-1} q \right) - \frac{3}{4} \overline{\partial} \left(\overline{u} q \right) + \dot{q}$$

where

$$I = c\overline{\partial} \left(\int \left\{ 3ik\partial^2 - 3k^2\partial - \frac{3}{4}iku \right\} e^{i\tau S} \mathbf{t}^{\sharp}\overline{\mu} \right)$$

We claim that

(3.34)
$$I = \frac{3}{4} \left\{ \overline{\partial} \left(u \partial \overline{\partial}^{-1} q \right) - \partial \left(u q \right) \right\}.$$

If so, then q solves the NV equation as claimed. To compute I, write $I = I_1 - I_2$ where

$$I_{1} = c\overline{\partial} \left(\int \left\{ 3ik\partial^{2} - 3k^{2}\partial \right\} e^{i\tau S} \mathbf{t}^{\sharp}\overline{\mu} \right),$$

$$I_{2} = c\overline{\partial} \left(\int e^{i\tau S} \mathbf{t}^{\sharp} \left\{ \frac{3}{4}iku \right\} \overline{\mu} \right).$$

Using (3.33) and (3.32) we may write

$$\begin{split} I_{1} &= \frac{3}{4}c\partial\left(\int e^{i\tau S}\left(ik\right)\mathbf{t}^{\sharp}q\overline{\mu}\right) \\ &= \frac{3}{4}c\partial\left(q\int\left(-\partial e^{i\tau S}\right)\mathbf{t}^{\sharp}\overline{\mu}\right) \\ &= -\frac{3}{4}c\partial\left(q\partial\left(\int e^{i\tau S}\mathbf{t}^{\sharp}\overline{\mu}\right) - q\int e^{i\tau S}\mathbf{t}^{\sharp}\partial\overline{\mu}\right) \\ &= -\frac{3}{4}\partial\left(q\partial\overline{\partial}^{-1}q\right) + \frac{3}{4}c\partial\left(q\overline{\partial}^{-1}\overline{\partial}\left(\int e^{i\tau S}\mathbf{t}^{\sharp}\partial\overline{\mu}\right)\right) \end{split}$$

where in the third line we used $u\partial v = \partial (uv) - v\partial u$. In the second term on the fourth line, we may use

(3.35)
$$\overline{\partial} e^{i\tau S} \partial \overline{\mu} = e^{i\tau S} \partial \left(\overline{\partial} - i\overline{k} \right) \overline{\mu}$$

and (3.33) to conclude that

$$\frac{3}{4}c\partial\left(q\overline{\partial}^{-1}\overline{\partial}\left(\int e^{i\tau S}\mathbf{t}^{\sharp}\partial\overline{\mu}\right)\right) = \frac{3}{16}c\partial\left(q\overline{\partial}^{-1}q\left(\int e^{i\tau S}\mathbf{t}^{\sharp}\overline{\mu}\right)\right)$$
$$= \frac{3}{16}\partial\left(q\overline{\partial}^{-1}\left(q\overline{\partial}^{-1}q\right)\right)$$

so that

$$I_{1} = -\frac{3}{4}\partial\left(q\partial\overline{\partial}^{-1}q\right) + \frac{3}{16}\partial\left(q\overline{\partial}^{-1}\left(q\overline{\partial}^{-1}q\right)\right)$$
$$= -\frac{3}{4}\partial\left(qu\right) + \frac{3}{16}\partial\left(q\overline{\partial}^{-1}\left(q\overline{\partial}^{-1}q\right)\right).$$

Similarly, we may compute

$$\begin{split} I_2 &= \frac{3}{4}c\overline{\partial}\left(u\int ik\ e^{itS}\mathbf{t}^{\sharp}\overline{\mu}\right) \\ &= \frac{3}{4}c\overline{\partial}\left(u\int\left(-\partial e^{i\tau S}\right)\ \mathbf{t}^{\sharp}\overline{\mu}\right) \\ &= -\frac{3}{4}c\overline{\partial}\left(u\partial\left(\int e^{i\tau S}\mathbf{t}^{\sharp}\overline{\mu}\right) - u\int e^{i\tau S}\mathbf{t}^{\sharp}\partial\overline{\mu}\right) \\ &= -\frac{3}{4}\overline{\partial}\left(u\overline{\partial}^{-1}\partial q\right) + \frac{3}{4}c\overline{\partial}\left(u\overline{\partial}^{-1}\left(\int \mathbf{t}^{\sharp}\overline{\partial}\left(e^{i\tau S}\partial\overline{\mu}\right)\right)\right). \end{split}$$

Using (3.35) again we find

$$I_{2} = -\frac{3}{4}\overline{\partial}\left(u\overline{\partial}^{-1}\partial q\right) + \frac{3}{16}\overline{\partial}\left(u\overline{\partial}^{-1}\left(q\overline{\partial}^{-1}q\right)\right)$$
$$= -\frac{3}{4}\overline{\partial}\left(u^{2}\right) + \frac{3}{16}\overline{\partial}\left(u\overline{\partial}^{1}\left(q\overline{\partial}^{-1}q\right)\right)$$

where we used (3.33) in the first line, and in the second line we used $u = \overline{\partial}^{-1} \partial q$. Hence

$$I_{1} - I_{2} = -\frac{3}{4}\partial(qu) + \frac{3}{4}\overline{\partial}(u^{2}) + \frac{3}{16}\partial(q\overline{\partial}^{-1}(q\overline{\partial}^{-1}q)) - \frac{3}{16}\overline{\partial}(u\overline{\partial}^{1}(q\overline{\partial}^{-1}q)).$$

Since $\partial q = \overline{\partial}u$ and $\overline{\partial}^{-1}\left(q\overline{\partial}^{-1}q\right) = \frac{1}{2}\left(\overline{\partial}^{-1}q\right)^2$ we can conclude that the second line is zero and (3.34) holds. The conclusion now follows.

4. Special Solutions

There are various powerful methods to find solutions of nonlinear evolution equations, most notably the inverse scattering method. However, the inverse scattering method is not readily useful for finding closed-form solutions to the NV equation, and so techniques including Hirota's method and the extended mapping approach (EMA) are presented here to construct closed-form solutions of several types of solitons. We begin by explaining the close connection between plane-wave solutions to NV and solutions to the KdV equation and present evolutions of KdV ring-type solutions. Although the KdV ring-type solition is not of conductivity type, the scattering transform is computed in Section 4.3, and the numerical results provide evidence of the presence of an exceptional circle.

4.1. KdV-type Solutions. Consider the NV equation (3.1) in the form

(4.1)
$$\dot{q} = -\frac{1}{4}q_{xxx} + \frac{3}{4}q_{xyy} + \frac{3}{4}\operatorname{div}((q-E)\mathbf{u})$$

where $u = u_1 + iu_2$ and $\mathbf{u} = (u_1, u_2)$, and the auxiliary equation $\overline{\partial} u = \partial q$ as

(4.2)
$$\begin{cases} (u_1)_x - (u_2)_y &= +q_x \\ (u_2)_x + (u_1)_y &= -q_y \end{cases}$$

As in [13] we use a FFT-based method to solve the equations on the square $-L \le x, y \le +L$ with periodic boundary conditions.

To examine the linear contributions we introduce a function

$$q(t, x, y) = \exp(i\left(\xi x + \eta y\right))$$

Then the $\overline{\partial}$ equation (4.2) is solved by

$$u_1(t, x, y) = \frac{\xi^2 - \eta^2}{\xi^2 + \eta^2} \exp(i(\xi x + \eta y))$$

$$u_2(t, x, y) = \frac{-2\xi\eta}{\xi^2 + \eta^2} \exp(i(\xi x + \eta y)),$$

and thus the linear part of the NV equation (4.1)

$$\dot{q} = -\frac{1}{4} \frac{\partial^3}{\partial x^3} q + \frac{3}{4} \frac{\partial^3}{\partial x \partial y^2} q + E \frac{3}{4} \nabla \cdot \mathbf{u}$$

is transformed into a elementary linear ODE for the Fourier coefficient c(t)

$$4\frac{d}{dt}c(t) = i\left(\xi^3 - 3\,\xi\,\eta^2\right)\,\left(1 - \frac{3\,E}{\xi^2 + \eta^2}\right)\,c(t).$$

Assuming a Fourier approximation of the solutions

$$q(t, x, y) = \sum_{j,k=0}^{N-1} c_{j,k}(t) \, \exp(ipi \, (k \, x + j \, y)/L)$$

this leads to a coupled system of ODEs for the Fourier coefficients $c_{j,k}(t)$. We use a Crank–Nicolson scheme for the linear part of NV and an explicit method for the nonlinear contribution div $(q \mathbf{u})$. For details see [13,14].

There is a close connection between plane wave solutions to NV and solutions to KdV (see [13]):

REMARK 4.1. Assume the solutions to NV are planar waves

$$q(t,s) = q(t,x,y) = q(t,n_1s,n_2s)$$

$$u_i(t,s) = u_i(t,x,y) = u_i(t,n_1s,n_2s)$$

for some direction vector $\vec{n} = (n_1, n_2) = (\cos(\alpha), \sin(\alpha))$. Then the bounded solutions to the $\overline{\partial}$ equation (4.2) are given by

$$u_1(t,s) = +(n_1^2 - n_2^2) q(t,s) + c_1$$

$$u_2(t,s) = -(2n_1 n_2) q(t,s) + c_2$$

for arbitrary constants c_1, c_2 , and the NV equation (4.1) reduces to an equation similar to the KdV equation,

(4.3)
$$\frac{4}{\kappa}q_t = -q''' + 6 q q' + \frac{3\beta}{\kappa}q'$$

with $\kappa = \cos(3\alpha)$ and $\beta = -\kappa E + c_1 n_1 + c_2 n_2$. If v(t, s) denotes a solution to the standard KdV equation

(4.4)
$$\dot{v}(t,x) = -v'''(t,x) + 6v(t,x)v'(t,x).$$

we obtain explicit solutions to NV by

$$q(t,s) = v\left(\frac{\kappa}{4}t, s+k_1t\right) - k_2 = v\left(\frac{\kappa}{4}t, s+\frac{3}{4}(c_1n_1+c_2n_2)t\right) + \frac{E}{2}$$

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For the special case $c_1 = c_2 = 0$ we find

$$q(t, n_1 s, n_2 s) = q(t, s) = v\left(\frac{\kappa}{4}t, s\right) + \frac{E}{2}$$

Thus we can relate all solutions to KdV as planar solutions to NV, with different speeds of propagation depending on the direction of the plane wave.

EXAMPLE 4.2. Based on the above remark we have an exact solution of the NV equation at zero energy given by

$$q(t, x, y) = -2c \cosh^{-2}(\sqrt{c}(x - ct)).$$

This solution is unstable with respect to perturbations periodic in y direction with period $\frac{2\pi}{k\sqrt{c}}$ for 0.363 < k < 1, see [13]. This is confirmed by numerical evolution of the NV equation using the above spectral method.

EXAMPLE 4.3. A KdV ring initial condition. Using $r = \sqrt{x^2 + y^2}$ we choose a radially symmetric initial value

$$q_0(x,y) = q(0,x,y) = f(r) = -\frac{1}{2} \cosh^{-2}(\frac{1}{2}(r-20)) < 0$$
.

This corresponds to a solution of equation (4.3) in the radial variable r, a KdV ring with radius 20.

Using the argument in [56, Appendix B] (based on [53]) we may consider q_0 as a non-positive deviation from 0, which is a potential of conductivity type. Thus q_0 is subcritical and consequently not of conductivity type. In [53] Murata classifies general solutions of the Schrödinger equation $\Delta u = q u$, For our special case we have an elementary proof for the required result. For q(0, x, y) to be of conductivity type we would need a positive, radially symmetric function u such that $\Delta u = q u$ or in radial coordinates (r u'(r))' = r f(r) u(r). Since the function is radially symmetric we use u'(0) = 0 and an integration leads to

$$r \, u'(r) = 0 + \int_0^r s \, f(s) \, u(s) \, ds < 0$$

Thus u'(r) is negative and r u'(r) is decreasing. Consequently we have a constant $C = -r_0 u'(r_0) > 0$ and for all $r \ge r_0$ we conclude $u'(r) \le -C/r$. This implies

$$u(r) = u(r_0) + \int_{r_0}^r u'(s) \, ds \le u(r_0) - \int_{r_0}^r \frac{C}{s} \, ds = u(r_0) - C\left(\ln(r) - \ln(r_0)\right)$$

and for r large enough this is in contradition to u(r) > 0 and thus $q_0(x, y)$ is not of conductivity type.

We solve the NV equation at zero energy E = 0 with the above initial condition. Based on the speed profile with the angularly dependent speed factor $\kappa = \cos(3 \alpha)$ from Remark 4.1, one expects that the initially circular shape will be deformed and its shape will be more triangular at later times. This is confirmed by Figure 2, which shows graphs of -q(t, x, y) at different times t.

To examine the possible blowup of the solution at a finite time we ran the algorithm based on the spectral method on a domain $-50 \le x, y \le +50$ with Fourier grids of sizes 1024×1024 , 2048×2048 and 4096×4096 . With time steps dt = 0.01 and dt = 0.001 we examined the solution and its L_2 norm. In all cases the solution either blew up at times just beyond t = 38 or displayed a sudden occurrence of sizable noise. For a final decision of a blow up time the exact shape and size of the spikes in Figure 2 have to be examined carefully.

Letting the KdV ring initial condition evolve for negative times, we observed the same solutions as for positive times, but rotated by in the spatial plane by 60° . We observed blowup at times smaller that t = -38.



FIGURE 2. Evolution of a KdV ring by NV at zero energy

EXAMPLE 4.4. With the initial value of the KdV ring in Example 4.3, we evolve the solution by the NV equation (4.1) and (4.2) with a positive energy E = 1/8. Time snapshots of the evolution are plotted in Figure 3. The initial dynamics are comparable to the previous example at zero energy: three spikes appear and grow rapidly in size, but as time progresses, these spikes decay in amplitude, and separate from the previous KdV ring, and a new triple of spikes appears. The process is repeated. Observe that the solution exists at least until time t = 100, also confirmed by the graph of the L_2 norm of the solution as function of time in Figure 4.

4.2. Closed-form Solutions. Most, if not all, soliton equations admit traveling waves solutions that involve the hyperbolic secant function, which can be



FIGURE 3. Evolution of a KdV ring by NV at positive energy



FIGURE 4. L_2 norm of the NV solution at positive energy

written in terms of the hyperbolic tangent function. Moreover, the hyperbolic tangent function is a solution to the Riccati equation, $\phi' = l_0 + \phi^2$, for $l_0 < 0$ for certain initial conditions. The ubiquity of the hyperbolic functions as traveling wave solutions naturally leads to the idea of expansion methods for solving soliton equations.

Solutions in the literature to the NV equation include the solutions from the inverse scattering transform [44–46, 64], the classic hyperbolic secant and cnoidal solutions [60], and rational solutions derived using Darboux transformations that lead to finite time blow–up (see [88] and references therein). In this section we present new solutions to the NV equation using Hirota's bilinear method and the Extended Mapping Approach (EMA). New multiple traveling wave solutions using the Modified Extended Tanh-Function Method can be found in [14]. This approach results in closed-form solutions, most of which contain singularities. We note that the solutions found by Hirota's method are plane-wave solutions, that is, KdV-type solutions, while the EMA-derived solutions are not.

4.2.1. *Hirota's Method.* Following the pioneering work of Hirota [**31**], multisoliton solutions can be derived using *Hirota's bilinear method.* This method yields soliton solutions as a sum of polynomials of exponentials and was used in [**94**] to find multisoliton solutions to the Nizhnik-Novikov-Veselov equation

$$\dot{q} = -aq_{xxx} + bq_{yyy} - 3a(qu_1)_x - 3b(qu_2)_y q_x = (u_1)_y q_y = (u_2)_x$$

The main idea is to reduce the nonlinear equation to a bilinear form through a transformation involving the logarithmic function. To express the wave velocity c in terms of the dispersive coefficients, assume u is a plane wave solution with $k_1 = k_2 = k$, $u = e^{kx+ky-ct}$, and substitute u into (2.1). This results in $c = -k^3/2$. Under the transformations $q = R(\ln(f))_{xx}$, $v = R(\ln(f))_{xy}$, and $w = R(\ln(f))_{yy}$ where $f(x, y, t) = 1 + Ce^{kx+ky+\frac{k^3}{2}t}$ and C is an arbitrary constant, one can algebraically solve for R to find a bilinear form (one finds R = 2). This method results in the soliton solution

(4.5)
$$q(x,y,t) = u_1(x,y,t) = u_2(x,y,t) = \frac{2Ck^2e^{k(2x+2y+k^2t)/2}}{\left(1 + e^{k(2x+2y+k^2t)/2}\right)^2}.$$

Choosing

$$f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1 + \theta_2},$$

where $\theta_i = k_i x + k_i y + \frac{1}{2} k_i^3 t$, i = 1, 2, in the logarithmic transformations above results in the two-soliton solution with a_{12} given in terms of k_1 and k_2 by $a_{12} = (k_1 - k_2)^2/(k_1 + k_2)^2$

(4.6)
$$q(x, y, t) = \frac{2\left(k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + (k_1 - k_2)^2 e^{\theta_1 + \theta_2}\right)}{1 + e^{\theta_1} + e^{\theta_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{\theta_1 + \theta_2}} - \frac{2\left(k_1 e^{\theta_1} + k_2 e^{\theta_2} + \frac{(k_1 - k_2)^2}{k_1 + k_2} e^{\theta_1 + \theta_2}\right)^2}{\left(1 + e^{\theta_1} + e^{\theta_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{\theta_1 + \theta_2}\right)^2}.$$

The evolution of the two-soliton solution is plotted in Figure 5. Further details and a three-soliton solution are found in [14].



FIGURE 5. Time snapshots of the evolution of a 2-soliton solution derived Hirota's bilinear method.

4.2.2. Extended Mapping Approach. The extended mapping approach (EMA) was presented formally by Zheng [97] and extends results by Lou and Ni [47]. The method is designed to find mappings between nonlinear PDE's. In this approach, q, v, w are expanded in terms of a function ϕ_i that satisfies the Riccati equation

$$\frac{d\phi}{dR} = \ell_0 + \phi^2,$$

where R = R(x, y, t). Thus, $q(x, y, t) = \sum_{i=0}^{n} a_i \phi^i$, $v(x, y, t) = \sum_{i=0}^{m} b_i \phi^i$, and $w(x, y, t) = \sum_{i=0}^{k} c_i \phi^i$, where the values of n, m and k are determined by balancing the highest order derivative terms with the nonlinear terms of the PDE. The method is described nicely in [78]. The balancing method results in n = m = k = 2. Substituting these expansions into the NV equation and equating coefficients of the resulting polynomial in ϕ results in a system of thirteen PDE's from which

we need to solve for the coefficients $a_i(x, y, t), b_i(x, y, t)$, and $c_i(x, y, t), i = 1, 2$. Using a separation technique for R, namely, R(x, y, t) = p(x, t) + q(y, t) results in $sech^2$ solutions, static (time-independent) solutions, and breather-type solutions of the NV equation. Further details, including the choices of ϕ are found in [14]. A time-independent solution is given by

$$q(x, y, t) = \frac{-1728 y^6 + (-96 + 1728 C)y^4 + (-40 + 288 C) y^2 - 36 C + 5}{432 y^4 - 36 y^2}$$
$$- 4 \tanh(x + y^2) + (2 + 8 y) \tanh^2(x + y^2)$$
$$v(x, y, t) = \frac{144 y^4 + (-12 + 432 C)y^2 - 36 C + 5}{36 y^2}$$
$$+ 4 \tanh(x + y^2) + (2 - 8 y) \tanh^2(x + y^2)$$
$$w(x, y, t) = 8 y - 8 y \tanh^2(x + y^2),$$

where C is an arbitrary constant. See Figure 6 for a plot of the solution q with C = 1.



FIGURE 6. A static solution to the NV equation

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Breather solutions are solutions with a type of periodic back-and-forth motion in time. One particular breather solution from [14] is

$$q(x, y, t) = \frac{-1728 y^6 + (-96 + 1728 C) y^4 + (-40 + 288 C) y^2 - 36 C + 5}{432 y^4 - 36 y^2} - 4 \tanh(1 + x + y^2 + 4 \cos t) + (2 + 8 y^2) \tanh^2(1 + x + y^2 + 4 \cos t)$$

$$v(x, y, t) = \frac{-192 \sin(t) y^2 + 144 y^4 + (-12 + 432 C) y^2 - 36 C + 5}{36 y^2} + 4 \tanh(1 + x + y^2 + 4 \cos t) + (2 - 8 y^2) \tanh^2(1 + x + y^2 + 4 \cos t)$$

$$w(x, y, t) = 8y - 8y \tanh^2(1 + x + y^2 + 4\cos t).$$

Several time snapshots are shown in Figure (7). For multisoliton solutions the reader is referred to [14].

For multisoliton solutions the reader is referred to [14]

4.3. Scattering transform of the ring soliton at time zero. In this section we compute numerically the scattering transform of the KdV ring soliton discussed in example 4.3 and illustrated in Figure 2. Since the initial potential is supercritical, we expect the scattering transform to have a singularity. Since the initial potential is real-valued and rotationally symmetric in the z-plane, also the scattering transform is real-valued and rotationally symmetric in the k-plane. See [56, Appendix A] for a proof. Therefore it is enough to compute $\mathbf{t}(k)$ only for real and positive k.

We use both the LS and the DN methods described in Section 3.2 and compare the results to verify accuracy. The values in the range $0.1 \le |k| \le 4$ are reliable as the results of both NV and LS methods closely agree there. However, the DN method does not give reliable results for |k| > 4.

To assure accuracy for |k| > 4, we compare the results of the LS method with two different grids in the z-domain. The coarser grid has 4096 × 4096 points, and the finer grid has 8192 × 8192 points. The coarser grid is not a subset of the finer grid. We remark that both of these grids are significantly finer than those we typically use for computing scattering transforms for conductivity-type potentials. Due to high memory requirements, we used a liquid-cooled HP Z800 Workstation with 192 GB of memory. Even with that powerful machine, the evaluation of one point value using the finer grid takes more than 11 hours. The LS method gives closely matching results in the regions $3 \le |k| \le 5$ and |k| > 9.

Figure 8 shows the profile of the scattering transform. In the interval 5 < |k| < 9 the numerical computation does not converge, resulting either in inaccurate evaluation of the point values of the scattering transform or in complete failure of the algorithm due to using up all the memory. We suspect that the observed numerical divergence arises from the existence of at least one exceptional circle in the interval 5 < |k| < 9.



FIGURE 7. Time snapshots of a breather solution derived using the EMA.

5. Zero-energy exceptional points

The inverse scattering method for the solution of the Novikov-Veselov equation is based on the complex geometrical optics (CGO) solutions ψ of the Schrödinger equation (2.8). The function $\psi(z,k)$ is asymptotically close to the exponential function e^{ikz} in the sense of formula (2.9); the point is that ψ can be used to define a nonlinear Fourier transform $\mathbf{t}(k)$ specially designed for linearizing the NV equation. See diagram (1.5) above.

However, there is a possible difficulty in using ψ and **t**. Even in the case of a smooth and compactly supported potential $q \in C_0^{\infty}$, there may exist complex numbers $k \neq 0$ for which equation (2.8) does not have a unique solution satisfying the asymptotic condition (2.9). Such k are called *exceptional points* of q. It is shown in [56] that that exceptional points of rotationally symmetric potentials come in circles centered at the origin and that the scattering transform has a strong



FIGURE 8. Profile of the scattering transform $\mathbf{t}(k)$ of the KdV ring soliton for k ranging in the positive real axis. The computation was unstable in the region 5 < |k| < 9, suggesting the presence of an exceptional circle.

singularity at the circles. The singularity prevents any currently understood use of the inverse nonlinear Fourier transform in the diagram (1.5). It seems safe to assume that the situation becomes only worse for more general potentials.

What is the connection between exceptional points and dynamics of solutions of the Novikov-Veselov equation? For example, does the absence of exceptional points in the initial data ensure smooth NV evolution? Do exceptional points perhaps correspond to lumps or solitons or finite-time blow-ups? Such a conjecture was presented already in [9, page 27], but the question is still open.

This section is devoted to a computational experiment illustrating exceptional points of a parametric family of rotationally symmetric potentials. The example clarifies the relationship between exceptional points and the trichotomy supercritical/critical/subcritical presented in Definition 1.1.

Take a radial C_0^2 function w(z) = w(|z|) as shown in Figure 9. A detailed definition of w is given in [55, Section 5.1]. Define a family of potentials by $q_{\lambda} = \lambda w$, parameterized by $\lambda \in \mathbb{R}$. Now the case $\lambda = 0$ gives $q_0 \equiv 0$, which is a critical potential since it arises as $q_0 = \sigma^{-1}\Delta\sigma$ with the positive function $\sigma \equiv 1$. From Murata [52] we see that $\lambda < 0$ gives a supercritical potential and $\lambda > 0$ gives a subritical potential. See [56, Appendix B] for details.

We use the DN method described in Section 3.2 to compute the scattering transforms of the potentials q_{λ} for the parameter λ ranging in the interval [-25, 5]. Since each potential $q_{\lambda}(z)$ is real-valued and rotationally symmetric in the z-plane, also the scattering transform is real-valued and rotationally symmetric in the k-plane. See [56, Appendix A] for details. Therefore it is enough to compute $\mathbf{t}(k)$ only for k ranging along the positive real axis. In Figure 11 we show the result of the computation as a two-dimensional grayscale image.

It is known that critical potentials do not have nonzero exceptional points; see [56, 58]. Thus there are no singularities in Figure 11 for $\lambda = 0$ (actually in this simple example we have $\mathcal{T}q_0 \equiv 0$). Furthermore, a Neumann series argument shows that for a fixed λ there exists such a positive constant $K = K(\lambda)$ that there are no exceptional points for q_{λ} satisfying |k| > K. See the analysis in [58, above formula (1.12)].



FIGURE 9. Left: rotationally symmetric test function w(z). Right: the profile w(|z|).

According to [54], subcritical potentials do not have nonzero exceptional points. Thus there are no singularities in Figure 11 for parameter values $\lambda \geq 0$. (We remark that the seemingly exceptional curves in the upper right corner of Figures 3 and 9 in [56] are due to deteriorating numerical accuracy for large positive values of λ . Those figures are trustworthy only for λ close to zero.)

For negative λ close to zero it is known from [56] that there is exactly one circle of exceptional points. The asymptotic form of that radius as a function of λ is calculated explicitly in [56].

For $\lambda \ll 0$ there is no precise understanding of exceptional points as of now; numerical evidence such as Figure 11 suggests that there may be several exceptional circles. Also, something curious seem to happen around $\lambda \approx -8$ and $\lambda \approx -20.5$; at present there is no explanation available.

See [56] for more zero-energy examples and [77] for analogous evidence of exceptional points at positive energies.

6. Open Problems

6.1. Applications of the NV Equation. The stationary NV equation [19] and the modified NV equation [83] have applications in differential geometry. Although the NV equation (1.1) is not known to be a mathematical model for any physical dynamical system, there has been some research in this direction for the dispersionless Novikov-Veselov (dNV) equation

(6.1)
$$q_{\xi} = (uq)_z + (\overline{u}q)_{\overline{z}}$$
$$u_{\overline{z}} = -3q_z.$$

Equation (6.1) was derived in [43] as the geometrical optics limit of Maxwell's equations in an anisotropic medium. The model governs the propagation of monochromatic electromagnetic (EM) waves of high frequency ω . In particular, they consider nonlinear media with Cole-Cole dependence [12] of the dielectric function and magnetic permeability on the frequency. Assuming slow variation of all quantities along the z axis, it is shown that Maxwell's equations reduce to (6.1) where $q = n^2$, the



FIGURE 10. Left column: profiles of rotationally symmetric potentials $q_{\lambda}(|z|)$ resulting from different values of λ . Right column: profiles of corresponding scattering transforms $t_{\lambda}(|k|)$. Note that negative values of λ lead to exceptional circles where the scattering transform is singular. See also Figure 11 which shows scattering transform profiles corresponding to more values of λ .



FIGURE 11. Profiles of rotationally symmetric scattering transforms. Lighter shade indicates larger values and darker shade smaller values. The horizontal axis is the parameter λ in the definition $q_{\lambda}(z) = \lambda w(z)$ of the potential. The vertical axis is |k|. There are curves along which a singular jump "from $-\infty$ to $+\infty$ " appears, indicated by an abrupt change from black to white. The k values at those curves are exceptional points. Note that the profiles corresponding to $\lambda = -10, -5, 0, 5$ are shown also in Figure 10.

refractive index, and ξ is a "slow" variable defined by $z = \omega^{\nu} \xi$. The phase of the electric field is governed by [43]

(6.2)
$$S_x^2 + S_y^2 = n^2(x, y, \xi)$$
$$S_\xi = \phi(x, y, \xi; S_x, S_y)$$

for a real-valued function ϕ , and the dNV heirarchy characterizes both the phase and refractive index.

In [8] hydrodynamic-type reductions of the dNV equation are presented, but the physical interpretation of these reductions are left for future work. Interesting open problems are whether the inclusion of the dispersion terms in the NV equation

models EM waves in a manner related to those derived for the dNV equation, and whether either NV or dNV serves as a physical model for any kind of hydrodynamic phenomenon.

6.2. Exceptional Sets and Large-Time Behavior for the NV Equation. Recall that the exceptional set of a potential q_0 is the set of all those $k \in \mathbb{C}$ for which there is not a unique NCGO solution $\mu(z, k)$. Here we discuss the relationship between: (1) the "size" of the exceptional set," (2) whether the potential q_0 is subcritical, critical, or supercritical (recall this trichotomy, Definition 1.1 from the introduction) and (3) whether the NV equation with initial data q_0 has a global solution. In these remarks, we will usually restriction attention to realvalued potentials q_0 belonging to the space $L^p_{\rho}(\mathbb{R}^2)$ for $\rho > 1$ and $p \in (1, 2)$ (see the remarks preceding Definition 1.3); one may think of potentials of order $\mathcal{O}(|z|^{-2-\varepsilon})$ as $|z| \to \infty$. Recall that, in the inverse scattering literature, critical potentials are usually referred to as "potentials of conductivity type."

To date, the only rigorous results on the size of exceptional sets for the zeroenergy NV equation are due to Nachman [58] and Music [54]. As explained above, Nachman showed that a potential is of conductivity type (or, equivalently, a critical potential as defined in the introduction, Definition 1.1) if and only if the exceptional set is empty and the scattering transform $\mathbf{t}(k)$ is $\mathcal{O}(|k|^{\varepsilon})$ as $|k| \to 0$ for some $\varepsilon > 0$. Music, extending Nachman's ideas and techniques, showed that a subcritical potential with sufficient decay at infinity has an empty exceptional set and characterized the singularity of the potential as $|k| \to 0$. Perry [67] showed that, if q_0 is a sufficiently smooth critical potential, the NV equation with initial data q_0 has a solution global in time. There is strong evidence to suggest that a similar result can be proved for the NV equation with subcritical initial data, based on the work of Music [54].

Thus, it remains to understand the singularities of the scattering transform for supercritical potentials. Examples due to Grinevich and Novikov [30] and Music, Perry and Siltanen [56] show that supercritical potentials may have circles of exceptional points. It is not known whether supercritical potentials *must* or *may* have exceptional points, nor is it known how to extend the inverse scattering formalism to potentials with nonempty exceptional sets. The following result due to Brown, Music, and Perry [10] gives an initial constraint on the size of exceptional sets for particularly nice potentials.

THEOREM 6.1. [10] Suppose that q is a real-valued measurable valued function with the property that $|q(z)| \leq C_1 \exp(-C_2|z|)$ for some constants C_1 and C_2 . Then the exceptional set of q consists at most of isolated points together with at most finitely many smooth curves with at most finitely many intersections.

To analyze the exceptional set, the authors define a renormalized determinant whose zero set is exactly the exceptional set. To describe it, let T_k is the integral operator

$$T_k\psi = \frac{1}{4}g_k * (q\psi).$$

The differential equation for $\mu(z, k)$, the NCGO solution, may be rewritten $\mu = 1 + T_k \mu$. Hence, uniqueness of solutions is equivalent to invertibility of $(I - T_k)$, and the exceptional set is exactly the set of points k for which $(I - T_k)$ fails to be invertible. It can be shown that T_k is a compact linear operator from L^p to itself

for any p > 2, and that T_k belongs to the so-called Mikhlin-Itskovich algebra of integral operators on L^p . For this reason we can apply the theory renormalized determinants due to Gohberg, Goldberg, and Krupnik [21] and define

$$\Delta(k) = \det_2(I - T_k)$$

where the determinant det₂ is the renormalized determinant. Brown, Music, and Perry show that this determinant is a real-analytic function of k for exponentially decaying potentials. It now follows from the Weierstrass preparation theorem that the zero set of $\Delta(k)$ is locally the zero set of a polynomial. Since the exceptional set is known to be closed and bounded, one can completely analyze the behavior of $\Delta(k)$ near the exceptional set using finitely many such local representations. It can be shown that $\Delta(k)$ is also real-valued, from which it follows that the zero set has the claimed form.

Theorem 6.1 opens up several areas for further investigation.

First, it would be of considerable interest to determine what additional data is needed to reconstruct a potential from $\mathbf{t}(k)$ when $\mathbf{t}(k)$ has point or line singularities.

Second, it would be very interesting to know whether singularities are always present for supercritical potentials, or whether, on the other hand, singularities are generically absent.

Third, our understanding of the NV equation and its dynamics would be greatly improved by connecting 'spectral' properties of the scattering transform (i.e., the nature of its singularities) to long-term behavior of solutions. The form of the time evolution for $\mathbf{t}(k)$ suggests that the 'trichotomy' of subcritical, critical, and supercritical potentials is invariant under the NV flow. It is known that critical initial data give rise to global solutions (see [68]), and there is strong evidence that the same is true of subcritical initial data. On the other hand, numerical experiments such as the ring soliton, Example 4.3, and analytical solutions such as those produced by Taimanov and Tsarev [85–88] strongly suggest that supercritical initial data lead to solutions of NV that blow up in finite time. It would be very interesting to obtain a rigorous proof that this is the case, and to analyze the nature of the blow-ups by inverse scattering methods.

Appendix A. Some Useful Analysis

In the direct scattering problem at zero energy, Faddeev's Green's function plays a critical role in elucidating properties of the CGO solutions that define the scattering transform. Recall that the normalized CGO solutions solve the equation

$$\partial (\partial + ik) \mu = (1/4)q\mu$$

and that Faddeev's Green's function is Green's function for the operator $\overline{\partial} (\partial + ik)$. On the other hand, the solid Cauchy transform is an inverse for the $\overline{\partial}$ operator with range in L^p functions for p > 2, and hence is a fundamental tool for solving the $\overline{\partial}$ problem that defines the inverse scattering transform. Finally, the Beurling transform is an integral operator which gives a meaning to the operator $\overline{\partial}^{-1}\partial$ that occurs in the definition of the nonlinearity in the NV equation. Here we collect some useful properties of these transforms and some essential estimates.

A.1. Faddeev's Green's Function at Zero Energy. We recall some key facts about Faddeev's Green's function g_k . We refer the reader to Siltanen's thesis [75] for details and references to the literature.

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Recall that g_k is defined by the formula

$$g_k(z) = \frac{1}{(2\pi)^2} \int e^{i\xi \cdot x} \frac{1}{\overline{\xi}(\xi+k)} \, dm(\xi)$$

where $z = x_1 + ix_2$, $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2$, $\xi = \xi_1 + i\xi_2$, and $\overline{\xi} = \xi_1 - i\xi_2$. By the Hausdorff-Young inequality, $g_k \in L^p$ for any p > 2. In fact, the estimate

$$\left\|g_k\right\|_p \le C_p \left|k\right|^{-2/p}$$

(see Siltanen [75], Theorem 3.10) holds for any $k \neq 0$. It is important to note that

$$g_k(z) = g_1(kz).$$

The following large-x asymptotic expansion of $g_1(x)$ is proved in [75], Theorem 3.11.

LEMMA A.1. Let $z = x_1 + ix_2$ with $z \neq 0$ and $x_1 > 0$. For any integer $N \ge 0$,

(A.1)
$$g_1(z) = -\frac{1}{4\pi} \sum_{j=0}^N \left[\frac{j!}{(iz)^{j+1}} - e^{-2ix_1} \frac{j!}{(-i\overline{z})^{j+1}} \right] + E_N(z)$$

where

(A.2)
$$|E_N(z)| \le \frac{(N+1)! 2^{(N+1)/2}}{\pi |z|^{N+2}}$$

Since $g(-x_1 + ix_2) = \overline{g_1(x_1 + ix_2)}$, similar formulas hold for $x_1 < 0$.

REMARK A.2. Since the error estimate (A.2) does not depend on the condition $x_1 > 0$, and since $g_1(z)$ is continuous, we can conclude that the expansion (A.1) remains valid for $z \neq 0$ and $\operatorname{Re}(z) = 0$.

Now consider $g_k(z) = g_1(kz)$. Since $\operatorname{Re}(kz) = \frac{1}{2}(kz + \overline{k}\overline{z})$, we immediately obtain:

LEMMA A.3. Let $z = x_1 + ix_2$ and $k \in \mathbb{C}$. For any integer $N \ge 0$, the expansion

(A.3)
$$g_k(z) = -\frac{1}{4\pi} \sum_{j=0}^{N} \left[\frac{j!}{(ikz)^{j+1}} - e^{-i(kz+\overline{k}\overline{z})} \frac{j!}{(-i\overline{k}\overline{z})^{j+1}} \right] + E_N(kz)$$

holds, where

$$|E_N(kz)| \le C_N |kz|^{-(N+2)}$$

A.2. The Cauchy Transform and the Beurling Operator. Following [3], chapter 4, we study the Cauchy transform and the Beurling operator through the logarithmic potential associated with Poisson's equation in two dimensions. For $\varphi \in C_0^{\infty}(\mathbb{R}^2)$, we may define the logarithmic potential

$$(L\varphi)(z) = \frac{2}{\pi} \int \log|z - z'| \ \varphi(z') \ dm(z')$$

which has the property

$$\partial \overline{\partial} \left(L \varphi \right) = \varphi.$$

Associated to L are the Cauchy transform,

$$(P\varphi)(z) = \frac{\partial}{\partial z} (L\varphi)(z),$$

the transform

$$\left(\overline{P}\varphi\right)(z) = \frac{\partial}{\partial \overline{z}}(L\varphi)(z),$$

and the Beurling transform

$$(\mathcal{S}\varphi)(z) = \frac{\partial^2}{\partial z^2}(L\varphi)(z).$$

From these definitions and (3.2), it is easy to see that

(A.4)
$$P \circ \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}} \circ P = I$$

where I is the identity on $\mathcal{C}_0^{\infty}(\mathbb{R}^2)$, and

(A.5)
$$S\left(\frac{\partial\varphi}{\partial\overline{z}}\right) = \frac{\partial\varphi}{\partial\overline{z}}$$

We have

$$(P\varphi)(z) = \frac{1}{\pi} \int \frac{1}{z - z'} \varphi(z') \ dm(z'),$$

$$\left(\overline{P}\varphi\right)(z) = \frac{1}{\pi} \int \frac{1}{\overline{z - z'}} \varphi(z) \ dm(z'),$$

and

(A.6)
$$(\mathcal{S}\varphi)(z) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \left(\int_{|z-z'| > \varepsilon} \frac{1}{(z-z')^2} \varphi(z') \ dm(z') \right).$$

The following estimates on P extend the Cauchy transform to L^p spaces and are standard consequences of the Hardy-Littlewood-Sobolev and Hölder inequalities (see Vekua [92] or [3], §4.3). They are used to prove existence and uniqueness of solutions to the $\overline{\partial}$ problem that defines the inverse problem.

LEMMA A.4. (i) For any
$$p \in (2, \infty)$$
 and $f \in L^{2p/(p+2)}(\mathbb{R}^2)$,
(A.7) $\|Pf\|_p \leq C_p \|f\|_{2p/(p+2)}$.

(ii) For any p,q with $1 < q < 2 < p < \infty$ and any $f \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, the estimate

(A.8)
$$||Pf||_{\infty} \le C_{p,q} ||f||_{L^p \cap L^q}$$

holds. Moreover, P is Hölder continuous of order (p-2)/p with

(A.9)
$$|(Pf)(z) - (Pf)(w)| \le C_p |z - w|^{(p-2)/p} ||f||_p.$$

(iii) If $v \in L^{s}(\mathbb{R}^{2})$ and q > 2 with $q^{-1} + 1/2 = p^{-1} + s^{-1}$, then for any $f \in L^{p}(\mathbb{R}^{2})$, (A.10) $\|P(vf)\|_{q} \leq C_{p,q} \|v\|_{s} \|f\|_{p}$.

REMARK A.5. Since $\mathcal{C}_0^{\infty}(\mathbb{R}^2)$ is dense in $L^p \cap L^q$ and $(Pf)(z) = \mathcal{O}(z^{-1})$ as $|z| \to \infty$ for any $f \in \mathcal{C}_0^{\infty}(\mathbb{R}^2)$, it follows from (ii) that if $f \in L^p \cap L^q$ for $1 , then <math>Pf \in \mathcal{C}_0(\mathbb{R}^2)$, the continuous functions that vanish at infinity.

REMARK A.6. Note that with s = q in (A.10) we have

$$\|P(vf)\|_{p} \leq C_{p} \|v\|_{2} \|f\|_{p}.$$

For any $f \in L^{2p/(p+2)}(\mathbb{R}^2)$, Lemma A.4 together with (A.4) imply that u = Pfsolves $\overline{\partial}u = f$ in distribution sense. Suppose, on the other hand, that $u \in L^p(\mathbb{R}^2)$ for some $p \in [1, \infty)$ and $\overline{\partial}u = 0$ in distribution sense. It follows that $\partial\overline{\partial}u =$ 0 in distribution sense, so that $u \in \mathcal{C}^{\infty}$ by Weyl's lemma. Thus, u is actually holomorphic, so u vanishes identically by Liouville's Theorem. From this fact and (A.4), we deduce:

LEMMA A.7. Suppose that $p \in (2, \infty)$, that $u \in L^p(\mathbb{R}^2)$, that $f \in L^{2p/(p+2)}(\mathbb{R}^2)$, and that $\overline{\partial}u = f$ in distribution sense. Then u = Pf. Conversely, if $f \in L^{2p/(p+2)}(\mathbb{R}^2)$ and u = Pf, then $\overline{\partial}u = f$ in distribution sense.

The following expansion for solutions of $\overline{\partial}u = f$ when f is rapidly decaying gives rise to the large-k asymptotic expansion for $\mu(z, k)$.

LEMMA A.8. Suppose that $p \in (2, \infty)$, that $u \in L^p(\mathbb{R}^2)$, that $f \in L^{2p/(p+2),N}(\mathbb{R}^2)$, and that $\overline{\partial} u = f$. Then

$$z^{N}\left[u(z) - \sum_{j=0}^{N-1} \frac{1}{z^{j+1}} \int \zeta^{j} f(\zeta) \ dm(\zeta)\right] \in L^{p}(\mathbb{R}^{2}).$$

PROOF. An immediate consequence of the estimate (A.7), Lemma A.7 and the formula

$$\frac{1}{z-\zeta} = \frac{1}{z} \sum_{j=0}^{N-1} \left(\frac{\zeta}{z}\right)^j + \frac{1}{z^N} \frac{\zeta^N}{z-\zeta}.$$

 \Box

REMARK A.9. If $f \in \mathcal{S}(\mathbb{R}^2)$ and depends smoothly on parameters, then the asymptotic expansion holds pointwise and is differentiable in the parameters.

The principal value integral (A.6) identifies S as a Calderón-Zygmund type integral operator. We have (see, for example, [3], §4.5.2):

LEMMA A.10. Suppose that $p \in (1, \infty)$. The operator S extends to a bounded operator from $L^p(\mathbb{R}^2)$ to itself, unitary if p = 2. Moreover, if $\nabla \varphi$ belongs to $L^p(\mathbb{R}^2)$ for $p \in (1, \infty)$, then $S(\partial \varphi) = \overline{\partial} \varphi$.

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