Contemporary Mathematics
Volume **635**, 2015
http://dx.doi.org/10.1090/conm/635/12678

Transverse instability of plane wave soliton solutions of the Novikov-Veselov equation

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Abstract. The Novikov-Veselov (NV) equation is a dispersive (2+1)-dimensional nonlinear evolution equation that generalizes the (1+1)-dimensional Korteweg-deVries (KdV) equation. This paper considers the instability of plane wave soliton solutions of the NV equation to transverse perturbations. To compute time evolutions numerically, a hybrid semi-implicit/spectral scheme was developed, applicable to other nonlinear PDE systems. Numerical simulations of the evolution of transversely perturbed plane wave solutions are presented using the spectral scheme. It is established that plane wave soliton solutions are not stable for transverse perturbations.

1. Introduction

The Novikov-Veselov (NV) equation for $u(z,t) = u(x,y,t)$ was introduced in the periodic setting by Novikov and Veselov [**28**] in the form

$$u_t = -\partial_z^3 u - \overline{\partial}^3 u + 3\partial_z(uf) + 3\overline{\partial}_z(u\bar{f}), \quad \text{where } \overline{\partial}_z f = \partial_z u, \tag{1.1}$$

with $\overline{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. It was derived algebraically from a Lax triple, and from this point of view is considered the most natural generalization of the KdV equation [**5**].

While physical applications of the NV equation are unknown, the dispersionless NV equation models the propagation of high frequency electromagnetic waves in certain nolinear media [**24, 25**]. It is also related to two other (2+1)-dimensional integrable systems which have been more widely studied, including the Davey-Stewartson II equation, which describes the complex amplitude of surface waves in shallow water, and was proved in [**31**] to be completely integrable.

Novikov and Veselov showed that the equation (1.1) is completely integrable [**28**]. Perry [**29**] gave an implementation of the completely integrable method for the Novikov-Veselov equation at zero energy that uses the inverse scattering map for the Davey-Stewartson equation together with Bogdonov's Miura transform.

2010 *Mathematics Subject Classification.* Primary 35C08, 35C07.

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If we consider real solutions $u(x, y, t)$ and let $f(x, y, t) = v(x, y, t) + i\,w(x, y, t)$, the NV equation has an equivalent representation in (x, y)-space:

$$4\,u_t = -u_{xxx} + 3\,u_{xyy} + 3\,(uv)_x + 3\,(uw)_y, \tag{1.2}$$

$$u_x = v_x - w_y, \tag{1.3}$$

$$u_y = -w_x - v_y. \tag{1.4}$$

This formulation is used throughout the paper since it is useful both for the numerics and the stability analysis. If the functions u, v and w are not dependent on y, the NV equation reduces to a KdV-type equation

$$4\,u_t + u_{xxx} - 6\,u\,u_x = 0, \qquad u = v, \qquad w = 0 \tag{1.5}$$

and admits soliton solutions of the form

$$u(x, y, t) = -2\,c\ \mathrm{sech}^2(\sqrt{c}(x - c\,t)), \tag{1.6}$$

$$v(x, y, t) = -2\,c\ \mathrm{sech}^2(\sqrt{c}(x - c\,t)), \tag{1.7}$$

$$w(x, y, t) = 0. \tag{1.8}$$

To facilitate the investigation of the qualitative nature of solutions to the NV equation, we present a version of a semi-implicit pseudo–spectral numerical scheme introduced by Feng et al [**13**] that solves the Cauchy problem for the NV equation. The method introduced by Feng et al was developed to compute solutions of 2-D nonlinear wave equations. It was used in [**2**] to compute evolutions of the modified KP equation with initial conditions chosen to be perturbed solitons. We have adapted it to solve systems of equations. We show that this method preserves the L_2 norm for the KdV type equation (1.5) and is considerably faster and requires less computer memory allocation than the finite difference scheme introduced in [**26**] for the solution of the Cauchy problem. The spectral method can be used to compute evolutions for a wide variety of initial conditions. Here, we take traveling wave solutions with small transverse perturbations as initial conditions in the Cauchy problem and compute their numerical evolution to study the nature of the instability that is proved in Section 4. The reader is referred to [**8**, **22**, **23**] for other examples of the use of spectral methods in soliton PDE's.

The problem of transverse stability of traveling wave solutions has been studied for many of the classic soliton equations including the KP equation [**2**, **4**, **7**, **19**], the Boussinesq equation [**4**], the ZK equation [**3**, **14**, **16**, **17**], and most notably, the KdV equation [**20**]. For the NV equation, we carry out a linear stability analysis by considering sinusoidal perturbations with wavefront perpendicular to the direction of propagation. In order to draw conclusions about the instability of soliton solutions, as well as approximate the growth rate, we apply the **K**-expansion method developed by Rowlands and Infeld. See [**18**] for an exposition of the method, [**3**] for an application of the method to the linearized Z-K equation, and [**6**] for an application in a soliton system modeling small amplitude long waves traveling over the surface of thin current-carrying metal film. The **K**-expansion method is based on the assumption that the nonlinear wave experiences a long-wavelength perturbation as well as a periodic change in wavelength [**18**].

In [**4**], the authors state the following conjecture: "It is interesting to speculate that when the expansion method introduced by Allen and Rowlands [**3**] is applied to any integrable system it will reveal that self similarity of the eigenfunctions will appear at each order, enabling one to write down the form of the exact solution.

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Alternatively this procedure may be seen as a necessary condition for integrability." The ZK equation and the equation in [**6**] are not integrable, and the ordinary perturbation analysis fails. So far this conjecture is supported by the KP and Boussinesq equations [**4**]. The results here support the conjecture by showing that only an ordinary perturbation analysis is needed for the NV equation.

The paper is organized as follows. In Section 2 we show that planar solutions to the NV equation must be a solution of a KdV-type equation. In Section 3 the semi-implicit pseudo-spectral method is presented, with its linear stability analysis in Section 3.1. The **K**-expansion method is used to establish the instability of traveling wave solutions of NV to transverse perturbations in Section 4. Numerical results are found in Section 5 and conclusions in Section 6.

2. From planar solutions of Novikov–Veselov to KdV

We examine planar solutions to the Novikov–Veselov equations (1.2)–(1.4), i.e. solutions that depend only on one spatial variable $s = (n_1, n_2) \cdot (x, y)$, moving in a direction given by the vector $\vec{n} = (n_1, n_2) = (\cos(\alpha), \sin(\alpha))$. We seek solutions of the form

$$\begin{aligned} u(t,s) &= u(t,x,y) = u(t, n_1\, s, n_2\, s) \\ v_i(t,s) &= v_i(t,x,y) = v_i(t, n_1\, s, n_2\, s) \quad \text{for } i = 1,2 \\ u'(t,s) &= \frac{\partial}{\partial s}\, u(t,s) = n_1 u_x(t,x,y) + n_2 u_y(t,x,y) \end{aligned}$$

The assumption that u, v_1 and v_2 are independent on $n_2\, x - n_1\, y$ is equivalent to

$$n_2 u_x - n_1 u_y = n_2\, \frac{\partial\, v_1}{\partial x} - n_1\, \frac{\partial\, v_1}{\partial y} = n_2\, \frac{\partial\, v_2}{\partial x} - n_1\, \frac{\partial\, v_2}{\partial y} = 0$$

As a consequence we obtain $u_x = n_1\, u'$ and $q_x = n_2\, u'$. The goal is to find a PDE for $u(t,s)$. For a given $u(t,s)$ the $\bar{\partial}$ equation

$$\begin{cases} \frac{\partial}{\partial x}\, v_1 - \frac{\partial}{\partial y}\, v_2 &= +u_x \\ \frac{\partial}{\partial x}\, v_2 + \frac{\partial}{\partial y}\, v_1 &= -u_y \end{cases}$$

translates to

$$\begin{cases} n_1\, v_1' - n_2\, v_2' &= +n_1\, u' \\ n_2\, v_1' + n_1\, v_2' &= -n_2\, u' \end{cases} \quad \Longrightarrow \quad \begin{aligned} v_1' &= (n_1^2 - n_2^2) \quad u' \\ v_2' &= -(2\, n_1\, n_2) \quad u' \end{aligned}$$

with the solutions

$$\begin{aligned} v_1(t,s) &= (n_1^2 - n_2^2)\; u(t,s) + c_1 \\ v_2(t,s) &= -(2\, n_1\, n_2)\; u(t,s) + c_2 \end{aligned}$$

The nonlinear expression in (1.2) leads to

$$\begin{aligned} \operatorname{div}(u\, \vec{v}) &= (n_1^2 - n_2^2)\, (u^2)_x - 2\, n_1\, n_2^2\, (u^2)_y + c_1 u_x + c_2 u_y \\ &= \left(2\, (n_1^2 - n_2^2)\, n_1 - 4\, n_1\, n_2^2\right)\; u\, u' + (c_1\, n_1 + c_2\, n_2)\, u' \\ &= 2\, \kappa\, u\, u' + \beta\, u' \end{aligned}$$

where

$$\beta = c_1\, n_1 + c_2\, n_2 \quad \text{and} \quad \kappa = \kappa(\alpha) = (n_1^2 - n_2^2)\, n_1 - 2\, n_1\, n_2^2 = \cos(3\, \alpha)$$

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The factor $\kappa = \cos(3\,\alpha)$ is further evidence for the threefold rotational invariance of the Novikov–Veselov equations. Now we examine the Novikov–Veselov equation

$$\begin{aligned} u_t &= -\frac{1}{4}u_{xxx} + \frac{3}{4}u_{xyy} + \frac{3}{4}\,\operatorname{div}(u\,\vec{v}) \\ &= -\frac{1}{4}\,n_1^3\,u''' + \frac{3}{4}\,n_1\,n_2^2\,u''' + \frac{6}{4}\,\kappa\,u\,u' + \frac{3}{4}\,\beta\,u' \\ &= -\frac{1}{4}\,\kappa\,u''' + \frac{6}{4}\,\kappa\,u\,u' + \frac{3}{4}\,\beta\,u' \end{aligned}$$

Thus, a planar solution of the Novikov–Veselov has to be a solution of the KdV-like equation

$$\frac{4}{\kappa}\,u_t = -u''' + 6\,u\,u' + \frac{3\,\beta}{\kappa}\,u' \tag{2.1}$$

The only essential modification is the contribution proportional to u' and thus it should not come as a surprise that the solutions are related. If $q(t,s)$ is a solution of the standard KdV equation

$$\dot{q}(t,x) = -q'''(t,x) + 6\,q(t,x)\,q'(t,x) \tag{2.2}$$

then

$$u(t,s) = q(\frac{\kappa}{4}\,t, s + \frac{3\,\beta}{4}\,t)$$

is a solution of the Novikov–Veselov equation (1.2), which can be verified as follows:

$$\begin{aligned} u_t &= \frac{\kappa}{4}q_t + \frac{3\,\beta}{4}q' \\ \frac{4}{\kappa}u_t + u''' - 6\,u\,u' - \frac{3\,\beta}{\kappa}\,u' &= q_t + \frac{3\,\beta}{\kappa}\,q' + q''' - 6\,q\,q' - \frac{3\,\beta}{\kappa}\,q' \\ &= q_t + q''' - 6\,q\,q' \;=\; 0 \end{aligned}$$

For a solution $q(t,s)$ of KdV choose constants k_1 and k_2 with

$$\frac{3\,\beta}{4} = k_1 + k_2\,\frac{3\,\kappa}{2}$$

and verify that

$$u(t,s) = q(\frac{\kappa}{4}\,t\,,\; s + k_1\,t) - k_2$$

is a solution of the Novikov–Veselov equation:

$$\begin{aligned} u_t &= \frac{\kappa}{4}q_t + k_1\,q' \\ \frac{4}{\kappa}\,u_t + u''' - 6\,u\,u' - \frac{3\,\beta}{\kappa}\,u' &= q_t + \frac{k_1\,4}{\kappa}\,q' + q''' - 6\,(q - k_2)\,q' - \frac{3\,\beta}{\kappa}\,q' \\ &= q_t + q''' - 6\,q\,q' + \left(\frac{k_1\,4}{\kappa} + 6\,k_2 - \frac{3\,\beta}{\kappa}\right)\,q' \\ &= q_t + q''' - 6\,q\,q' + \frac{4}{\kappa}\,\left(k_1 + k_2\,\frac{3\,\kappa}{2} - \frac{3\,\beta}{4}\right)\,q' \;=\; 0 \end{aligned}$$

Remarks:

- Consider the choice $\beta = k_1 = k_2 = 0$. Then $q(t,s) = u(\frac{\kappa}{4}\,t, s)$ is the KdV solution where the time scale is multiplied with $\kappa/4$. Since $\kappa = \kappa(\alpha) = \cos(3\,\alpha)$ the speed of the solution changes according to an angle dependent profile, as shown in Figure 1.
- The additive constant k_2 moves the KdV solution up or down in the graph.

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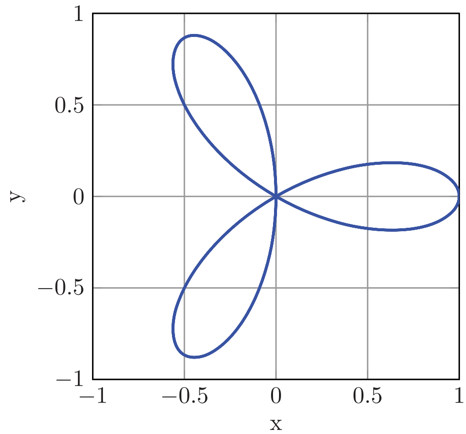

FIGURE 1. Speed profile for planar solutions of the Novikov–Veselov equation

- Replacing s by $s + k_1\, t$ (with $k_1 = \frac{3\,\beta}{4} - k_2 \frac{3\,\kappa}{2}$) corresponds to observing the KdV solution in a moving frame, where the frame moves with velocity $-k_1$.

3. A Pseudo-Spectral method for the solution of (2+1) nonlinear wave equations

To numerically solve (1.2)–(1.4) we use a semi–implicit leap–frog spectral method based on the method in [**13**]. We restrict ourselves to a finite spatial domain $\Omega = [0, W_x] \times [0, W_y]$ with periodic boundary conditions. Thus we work on a torus topology and for the numerical results we have to observe the traveling waves across the boundary. We must also choose the domain large enough for the problem at hand.

This method uses the Fast Fourier Transform (FFT) to compute the spatial evolution and a leapfrog scheme to compute the time-stepping. The numerical scheme can be summarized as follows:

The Fourier transform of (1.2)–(1.4) is

$$4\,\hat{u}_t = i\,\left(\xi^3 - 3\,\xi\,\eta^2\right)\,\hat{u} + 3\,i\,\xi\,\mathcal{F}[(uv)] + 3\,i\,\eta\,\mathcal{F}[(uw)], \tag{3.1}$$

$$\xi\,\hat{u} = \xi\,\hat{v} - \eta\,\hat{w}, \tag{3.2}$$

$$\eta\,\hat{u} = -\eta\,\hat{v} - \nu\,\hat{w}. \tag{3.3}$$

As usual, $\hat{u}$ refers to the Fourier transform F of u and is given by

$$\hat{u}_{p,q} = \mathcal{F}[u_{l,m}] = \sum_{l=0}^{L-1}\sum_{m=0}^{M-1} u_{l,m}\, e^{-i(\xi_p x_l + \eta_q y_m)},$$

$$u_{l,m} = \mathcal{F}^{-1}[\hat{u}_{p,q}] = \frac{1}{L\,M}\sum_{p=-L/2}^{L/2-1}\sum_{q=-M/2}^{M/2-1} \hat{u}_{p,q}\, e^{i(\xi_p x_l + \eta_q y_m)}.$$

The parameters L and M are the number of grid points in $[0, W_x] \times [0, W_y]$, respectively. They need to be powers of two in order to use the standard FFT. The spatial grid is defined by $(x_l, y_m) = (l\Delta x, m\Delta y)$ where $l = 0, \ldots, L-1$, and

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$m = 0, \dots, M-1$. The spectral variables are $(\xi_p, \eta_q) = (2\pi p/W_x, 2\pi q/W_y)$, with $p = -L/2, \dots, -1, 0, 1, \dots, L/2$, and $q = -M/2, \dots, -1, 0, 1, \dots, M/2$.

Equations (3.2) and (3.3) can be solved in terms of $\hat{u}$

$$\hat{v} = \frac{\xi^2 - \eta^2}{\xi^2 + \eta^2}\,\hat{u}, \qquad \hat{w} = \frac{-2\eta\,\xi}{\eta^2 + \xi^2}\,\hat{u} \tag{3.4}$$

Special attention has to be paid to the case $\xi^2 + \eta^2 = 0$. The corresponding Fourier coefficients represent the average values of the functions v and w.

Let $\vec{c}(t)$ be the vector with the LM Fourier coefficients of the solution $u(x,y,t)$. Then solving (3.2), (3.3) and computing $-3\,i\,(\xi\,\mathcal{F}[(uv)] + \eta\,\mathcal{F}[(uw)])$ may be written as one nonlinear function $\vec{F}(\vec{c})$. With the appropriate diagonal matrix $\mathbf{D}$ the NV equation (3.1) reads as

$$4\,\frac{d}{dt}\,\vec{c}(t) = \mathbf{D}\,\vec{c}(t) + \vec{F}(\vec{c}(t)) \tag{3.5}$$

For the time integration we use a symmetric three-level difference method for the linear terms, and a leapfrog method for the nonlinear terms. For the parameter $0 \le \theta \le 1$ we use a superscript to refer to the time iteration and the above leads to

$$\begin{aligned} \frac{4}{2\,\Delta t}\,(\vec{c}^{\,n+1} - \vec{c}^{\,n-1}) &= \theta\,\mathbf{D}\,(\vec{c}^{\,n+1} + \vec{c}^{\,n-1}) + (1 - 2\,\theta)\,\mathbf{D}\,\vec{c}^{\,n} + \vec{F}(\vec{c}^{\,n}) \\ (2 - \theta\,\Delta t\,\mathbf{D})\,\vec{c}^{\,n+1} &= (1 - 2\,\theta)\,\Delta t\,\mathbf{D}\,\vec{c}^{\,n} + (2 + \theta\,\Delta t\,\mathbf{D})\,\vec{c}^{\,n-1} \\ &\quad + \Delta t\,\vec{F}(\vec{c}^{\,n}) \end{aligned} \tag{3.6}$$

Thus we have an implicit scheme for the linear contribution and an explicit scheme for the nonlinear contribution. Since the matrix $\mathbf{D}$ is diagonal we do not have to solve a system of linear equations at each time step. For the special case $\theta = 1/2$ we obtain a Crank Nicolson scheme

$$(4 - \Delta t\,\mathbf{D})\,\vec{c}^{\,n+1} = +(4 + \Delta t\,\mathbf{D})\,\vec{c}^{\,n-1} + 2\,\Delta t\,\vec{F}(\vec{c}^{\,n})$$

For the three level method we need a separate method for the first time step. For the sake of simplicity we may choose $\vec{c}^{\,-1} = \vec{c}^{\,0}$. For computations with known solutions, e.g. (1.6), we use the known values of the solution at time $-\Delta t$.

To examine the efficiency of the spectral method, we compare it to the finite difference (FD) approach presented in [**26**]:

- The FD approach can be used for periodic boundary conditions and for Dirichlet type conditions. The spectral approach is perfectly suited for periodic boundary conditions.
- For both methods, a linear system of equations has to be examined at each time step. The corresponding matrix is of size $N^2 \times N^2$.
  - For the spectral method the matrix $\mathbf{D}$ is diagonal, and so we do not have to solve a system of equations. The computational effort is N^2.
  - For the FD approach the matrix has a semi-bandwidth $2\,N$. Thus, a sparse LU factorization is required to store approximately $4\,N^3$ numbers. For large N this is prohibitive. Thus, an iterative solver has to be used, such as `bicgstab`, with or without a preconditioner. The computational effort for each iteration is proportional to N^2, with a sizable constant of proportionality.
- For both algorithms the nonlinear contribution requires the use of a $2D$ FFT. For the FD approach this is accomplished with Dirchlet BC's on an

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enlarged domain. This is a sizable contribution to the total computation time.

- Both algorithms conserve the L_2 norm of the linear part and the stability behavior is similar.
- For both approaches a stability proof for the full, nonlinear equation is lacking. This should not come as a surprise, as the NV equations does have exponentially unstable solutions, examined in section 4.

Based on the above we conclude that the spectral method is a better approach for a problem with periodic boundary conditions. This is confirmed by a few test cases. For exact classical KdV soliton initial conditions, the spectral method reproduces the known exact solution with high accuracy on sizable time intervals.

3.1. Linear Numerical Stability Analysis. To gain insight into the stability of the spectral method, we include a linear stability analysis. The parameters determined here were used in the numerical experiments that follow. We examine the problem on a domain $\Omega = [0, W] \times [0, W] \subset \mathbb{R}^2$ and use periodic boundary conditions. Thus we examine the initial boundary value problem

$$\begin{aligned} 4\,u_t &= -u_{xxx} + 3\,u_{xyy} + 3\,\alpha\,(v_x + w_y), \\ u_x &= v_x - w_y, \qquad u_y = -w_x - v_y \end{aligned} \tag{3.7}$$

The case $\alpha = 0$ is the linearization of (1.2)–(1.4) about the zero solution. The utility of a linear stability analysis for a nonlinear system was addressed in [**13**] in the context of the KP and ZK equations, where they found that their results for the linear stability analysis were validated by numerical results and argued that while the analysis does not prove stability and convergence of the nonlinear scheme, the obtained stability conditions often suffice in practice. When implemented on a domain with periodic boundary condition (3.7) leads to

$$\begin{array}{ll} 4\,u_t = -u_{xxx} + 3\,u_{xyy} + 3\,\alpha\,(v_x + w_y), & (x,y)\times t \in \Omega\times\mathbb{R}, \\ u_x = v_x - w_y, \qquad u_y = -w_x - v_y & \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega, \\ u(x,y,t) = u(x+W_x, y, t), & (x,y)\times t \in \mathbb{R}^3, \\ u(x,y,t) = u(x, y+W_y, t), & (x,y)\times t \in \mathbb{R}^3, \end{array} \tag{3.8}$$

Using Fourier series and (3.4) this leads to

$$4\,\hat{u}_t \;=\; i\,(\xi^3 - 3\,\xi\,\eta^2)\,\hat{u} + 3\,i\,\xi\,\alpha\,\frac{\xi^2-\eta^2}{\eta^2+\xi^2}\,\hat{u} + 3\,i\,\eta\,\alpha\,\frac{-2\eta\,\xi}{\eta^2+\xi^2}\,\hat{u} = i\,\omega\,\hat{u} + i\,\frac{3\,\omega\,\alpha}{\eta^2+\xi^2}\,\hat{u},$$

where $\omega = \xi^3 - \xi\,\eta^2$. This is an ODE of the form

$$4\,\frac{d}{dt}\,u(t) = i\,\lambda\,u(t) + i\,\gamma\,u(t), \tag{3.9}$$

where $\lambda = \omega = \xi^3 - 3\,\xi\,\eta^2$ and $\gamma = \omega\,\frac{3\,\alpha}{\eta^2+\xi^2}$. Using the numerical scheme (3.6), this leads to

$$(2 - i\,\theta\,\lambda\,\Delta t)\,u^{n+1} = i\,(1-2\,\theta)\,\lambda\,\Delta t\,u^n + (2 + i\,\theta\,\lambda\,\Delta t)\,u^{n-1} + i\,\gamma\,\Delta t\,u^n,$$

or with

$$b_1 = 2 + i\,\theta\,\lambda\,\Delta t\, \in \mathbb{C}, \qquad b_2 = i\,((1-2\,\theta)\,\lambda + \gamma)\,\Delta t \in \mathbb{C}, \tag{3.10}$$

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to the iteration matrix

$$\left(\begin{array}{c} u^{n+1} \\ u^n \end{array}\right) = \mathbf{M} \left(\begin{array}{c} u^n \\ u^{n-1} \end{array}\right) = \left[\begin{array}{cc} \frac{b_2}{\overline{b_1}} & \frac{b_1}{\overline{b_1}} \\ 1 & 0 \end{array}\right] \left(\begin{array}{c} u^n \\ u^{n-1} \end{array}\right).$$

For the system to be stable we have to verify that the norm of the eigenvalues $z_{1,2}$ of $\mathbf{M}$ are less than or equal to 1. The characteristic equation is given by

$$\Phi_{NV}(z) = z^2 - \frac{b_2}{\overline{b_1}}\, z - \frac{b_1}{\overline{b_1}} = 0$$

Thus, we conclude $|z_i \cdot z_2| = |\frac{b_1}{\overline{b_1}}| = 1$ and stability of the solution is equivalent to $|z_{1,2}| = 1$, i.e., we have $|u^{n+1}| = |u^{n-1}|$. Using the explicit solution for quadratic equations and

$$b_2 \in i\,\mathbb{R}, \qquad |b_2| \leq 2\,|b_1|, \tag{3.11}$$

we find $2\,\overline{b_1}\, z_{1,2} = b_2 \pm \sqrt{b_2^2 + 4\,|b_1|^2}$ and $|2\,\overline{b_1}|^2\,|z_{1,2}|^2 = |2\,b_1|^2$. Consequently, $|z_{1,2}| = 1$. With (3.10), the stability condition (3.11) for the ODE (3.9) becomes

$$((1-4\,\theta)\,\lambda^2 + 2\,(1-2\,\theta)\,\lambda\,\gamma + \gamma^2)\,(\Delta t)^2 \quad \leq \quad 16.$$

For $\gamma = \alpha = 0$ and $\frac{1}{4} \leq \theta \leq 1$ the stability condition is satisfied, independent of the step size $\Delta t > 0$, and we have a stability result for the initial boundary value problem (3.8). This proves the following theorem.

THEOREM 1. *The numerical scheme* (3.6) *applied to the linearization of* (1.2)–(1.4) *about the zero solution is stable for* $\frac{1}{4} \leq \theta \leq 1$.

Proving stability for (3.8) with $\alpha \neq 0$ requires a few more computations. Using the expressions for λ and γ in (3.9) results in the stability condition

$$(1-4\,\theta)\; + 2\,(1-2\,\theta)\,\frac{3\,\alpha}{\eta^2+\xi^2} + \left(\frac{3\,\alpha}{\eta^2+\xi^2}\right)^2 \quad \leq \quad \left(\frac{4}{\omega\,\Delta t}\right)^2.$$

For $\theta = \frac{1}{2}$ we find the sufficient conditions

$$\begin{aligned} -2 + \left(\frac{3\,\alpha}{\eta^2+\xi^2}\right)^2 \leq \left(\frac{3\,\alpha}{\eta^2+\xi^2}\right)^2 \quad &\leq \quad \left(\frac{4}{(\xi^3 - 3\,\xi\,\eta^2)\,\Delta t}\right)^2 \\ (\Delta t)^2 \quad &\leq \quad \left(\frac{4\,(\xi^2+\eta^2)}{3\,\alpha\,(\xi^3 - 3\,\xi\,\eta^2)}\right)^2. \end{aligned}$$

Since $|\xi^3 - 3\,\xi\,\eta^2| \leq (\xi^2+\eta^2)\cdot\max\{|\xi|,\,|\eta|\}$ we have the sufficient condition for Δt

$$\Delta t \leq \frac{4}{3\,|\alpha|\,\max\{|\xi|,\,|\eta|\}}.$$

For the DFT with $\Delta x = \Delta y = \frac{W_x}{L}$ we find $\max\{|\xi|,\,|\eta|\} = \frac{\pi}{\Delta x}$, and thus we have the stability condition

$$\Delta t \leq \frac{4}{3\,|\alpha|\,\pi}\,\Delta x. \tag{3.12}$$

As a consequence we have a stability result for the initial boundary value problem (3.8) with $\alpha \neq 0$, Theorem 2.

THEOREM 2. *The numerical scheme* (3.6) *applied to* (3.8) *is stable for* $\theta = \frac{1}{2}$ *if the time step* Δt *satisfies condition* (3.12).

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REMARK 1. *The above results verify that the numerical scheme for the linear contributions is stable for small time steps for the Novikov–Veselov equation at nonzero energy $E \neq 0$, that is when equation (1.2) is replaced by*

$$4\,u_t = -u_{xxx} + 3\,u_{xyy} + 3\,(uv)_x + 3\,(uw)_y - E\,(v_x + w_y)$$

The divergence term $E\,(v_x + w_y)$ can either be integrated into the linear part of the scheme (i.e. in the matrix $\mathbf{D}$) or can be handled as part of the nonlinear contribution (i.e. in $\vec{F}(\vec{c})$).

4. Instability of traveling-wave solutions of the NV-equation to transverse perturbations

The **K**-expansion method presented by Allen and Rowlands considers long wavelength perturbations in the transversal direction. It was originally used to investigate the stability of solutions to the Zakharov–Kuznetsov equation, a non-integrable generalization of the KdV equation [**15**–**17**]. Since then the method has been applied to the KP equation and the modified ZK (mZK) equation [**27**]. This is the second application of the method to a system of PDE's, the first being by Bradley [**6**]. The **K**-expansion method is related to the method of Evans function [**1**, **9**–**12**] for determining soliton instability. The Evans function is an analytic function, the zeros of which correspond to the eigenvalues of the linearized operator. See [**21**] for explicit calculations of the Evans function in the stability analysis of pulses arising in nonlinear optics. Using the **K**-expansion method, we construct an eigenvalue with positive real part. This corresponds to a zero of the Evans function, also indicating instability.

4.1. The modified equations. To begin, we transform the system to move along with the soliton by using new independent and dependent variables. Using

$$(t, x, y) \mapsto \sqrt{c}\,(c\,t, x - c\,t, y) = (\tilde{t}, \tilde{x}, \tilde{y})$$

and

$$(u, v_1, v_2) \mapsto c\,(u, v_1, v_2)$$

or

$$\begin{aligned} u(t,x,y) &= c\,\tilde{u}(t\,c^{3/2}, (x - c\,t)\,c^{1/2}, y\,c^{1/2}) \\ v_1(t,x,y) &= c\,\tilde{v}_1(t\,c^{3/2}, (x - c\,t)\,c^{1/2}, y\,c^{1/2}) \\ v_2(t,x,y) &= c\,\tilde{v}_2(t\,c^{3/2}, (x - c\,t)\,c^{1/2}, y\,c^{1/2}), \end{aligned} \tag{4.1}$$

the NV equations (1.2)–(1.4) are transformed to

$$0 = 4u_t - 4u_x + u_{xxx} - 3u_{xyy} - 3(uv)_x - 3(uw)_y, \tag{4.2}$$

$$u_x = +v_x - w_y, \tag{4.3}$$

$$u_y = -w_x - v_y \tag{4.4}$$

For sake of a more readable notation we dropped the tildes on the new dependent and independent variables. The known solution

$$f_c(x - c\,t) = -2\,c\,\operatorname{sech}^2(\sqrt{c}\,(x - c\,t))$$

of the original system turns into a stationary solution of the modified system.

$$u_0(x) = v_0(x) = -2\,\operatorname{sech}^2(x), \qquad w_0(x) = 0$$

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Now we examine perturbed functions of the form

$$u(x,y,t) = u_0(x) + \epsilon\, f(x)\, e^{iky+\gamma t}, \tag{4.5}$$
$$v(x,y,t) = v_0(x) + \epsilon\, g(x)\, e^{iky+\gamma t}, \tag{4.6}$$
$$w(x,y,t) = w_0(x) + \epsilon\, h(x)\, e^{iky+\gamma t}. \tag{4.7}$$

Thus we examine periodic, transversal perturbations with exponential growth. For numerical purposes we may also work with a purely real formulation.

$$u(x,y,t) = u_0(x) + \epsilon\, f(x)\, \cos(k\,y)\, e^{\gamma t}, \tag{4.8}$$
$$v(x,y,t) = v_0(x) + \epsilon\, g(x)\, \cos(k\,y)\, e^{\gamma t}, \tag{4.9}$$
$$w(x,y,t) = w_0(x) + \epsilon\, h(x)\, \sin(k\,y)\, e^{\gamma t}. \tag{4.10}$$

Substituting these expressions into (4.2)–(4.4) and dropping terms proportional to ϵ^2 leads to (use $'$ for $\frac{d}{dx}$)

$$\begin{aligned} 0 &= 4\,\gamma\, f - 4\, f' + f''' + 3\, k^2\, f' \\ &\quad -3\,(u_0\, g)' - 3\,(v_0\, f)' - 3\, u_0\, i\, k\, h \end{aligned} \tag{4.11}$$
$$f' = +g' - i\, k\, h \tag{4.12}$$
$$i\, k\, f = -h' - i\, k\, g \tag{4.13}$$

This is a system of linear ordinary differential equations with non-constant coefficients. Use $h(x) = \frac{1}{i\,k}\,(g'(x) - f'(x))$ and the above can be written in the form

$$\begin{aligned} f'''(x) &= (-4\,\gamma + 3\, u_0'(x))\, f(x) + (4 - 3\, k^2)\, f'(x) + \\ &\quad +3\, u_0'(x)\, g(x) + 6\, u_0(x)\, g'(x) \end{aligned} \tag{4.14}$$
$$g''(x) = k^2\, g(x) + f''(x) + k^2\, f(x) \tag{4.15}$$

4.2. Known solutions. Since the $f_c(x - c\,t + s)$ is a solution of the original NV equation (1.2)–(1.4), we conclude that $(u_0(x-s),\, v_0(x-s),\, w_0(x-s))$ solve the system (4.2)–(4.4) and thus their derivatives must solve the linearized problem (4.11)–(4.13). Since $\frac{d}{dx}\, \operatorname{sech}^2(x) = -2\, \operatorname{sech}^2(x)\, \tanh(x)$ we find solutions

$$f(x) = g(x) = \operatorname{sech}^2(x)\, \tanh(x)\,, \qquad h(x) = 0 \tag{4.16}$$

for $k = \gamma = 0$ to the system (4.14)–(4.15). For $k = 1$ and $\gamma = 0$ we found

$$\begin{aligned} f(x) &= +\operatorname{sech}^3(x) \\ g(x) &= -\operatorname{sech}(x)\, \tanh^2(x) \\ h(x) &= -i\, \operatorname{sech}(x)\, \tanh(x) \end{aligned} \tag{4.17}$$

as a second, explicit solution of (4.11)–(4.13).

With the help of these solutions we will construct unstable solution to the Novikov-Veselov equations, as function of the parameters k and γ. Thus we need nonzero solutions of (4.14)–(4.15).

4.3. Locating unstable solutions. Using the function $u_0(x) = -2\, \operatorname{sech}^2(x)$, a matrix notation translates the high order system (4.14)–(4.15) into a system of

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first order ordinary differential equations.

$$\frac{d}{dx}\begin{pmatrix} f \\ f' \\ f'' \\ g \\ g' \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -4\,\gamma + 3\,u_0' & (4-3\,k^2) & 0 & 3\,u_0'(x) & 6\,u_0 \\ 0 & 0 & 0 & 0 & 1 \\ k^2 & 0 & 1 & k^2 & 0 \end{bmatrix}\begin{pmatrix} f \\ f' \\ f'' \\ g \\ g' \end{pmatrix}$$

We seek nontrivial solutions satisfying decay conditions as $x \to \pm\infty$. We use a combination of asymptotic analysis and numerical methods. It is necessary to consider large and small $|x|$ separately. We take the following approach:

- Choose a large value $M > 0$. For $|x| > M$ we use $u_0(x) \approx 0$ and an elementary asymptotic analysis to construct solutions satisfying the decay conditions.
- For $|x| \leq M$ an asymptotic analysis is not feasible, since the contributions $u_0(x)$ and $u_0'(x)$ in (4.14)–(4.15) have to be taken into account. Thus we use a numerical solver on the domain $-M \leq x \leq +M$ to construct a fundamental matrix $\mathbf{T}$ containing the information for the solutions of the linear system for $-M \leq x \leq M$.
- Then the above two results are combined to find criterion (4.23) for the existence of nontrivial solutions.

4.3.1. *Behavior of solution for large* $|x|$. For large values of $|x|$ we use $u_0(x) \approx 0$ and arrive at a decoupled system of linear differential equations with constant coefficients.

$$\frac{d}{dx}\begin{pmatrix} f \\ f' \\ f'' \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4\,\gamma & (4-3\,k^2) & 0 \end{bmatrix}\begin{pmatrix} f \\ f' \\ f'' \end{pmatrix} \tag{4.18}$$

$$\frac{d}{dx}\begin{pmatrix} g \\ g' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ k^2 & 0 \end{bmatrix}\begin{pmatrix} g \\ g' \end{pmatrix} + \begin{pmatrix} 0 \\ k^2\,f' + f'' \end{pmatrix} \tag{4.19}$$

The characteristic equation of the matrix in (4.18) is given by

$$\lambda^3 + (3\,k^2 - 4)\,\lambda + 4\,\gamma = 0, \tag{4.20}$$

and we will denote the eigenvalues λ_1, λ_2, and λ_3 of this system by p_1, p_2, and p_3, respectively. The inhomogeneous system in (4.19) leads to eigenvalues $\lambda_{4,5} = \pm k$. Using Cardano's formulas for zeros of polynomials of degree 3, we verify that (4.20) has three distinct real values if

$$4\,\gamma^2 + \frac{(3\,k^2-4)^3}{27} < 0.$$

For our domain of (k, γ) to be examined we may use

$$p_3 \leq 0 \leq p_1,\; p_2$$

and for small values of $|\gamma|$ we find

$$\lambda(\gamma) = \begin{cases} p_1 \approx 0 + \frac{4}{4-3\,k^2}\,\gamma \\ p_2 \approx +\sqrt{4-3\,k^2} - \frac{2}{4-3\,k^2}\,\gamma \\ p_3 \approx -\sqrt{4-3\,k^2} - \frac{2}{4-3\,k^2}\,\gamma \end{cases} \tag{4.21}$$

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with the solution of the form $f(x) \approx c_1 e^{p_1 x} + c_2 e^{p_2 x} + c_3 e^{p_3 x}$. We examine $\gamma \geq 0$, and for $x \to +\infty$ we require the solution $f(x)$ of (4.18) to be bounded, and thus

$$f(x) \approx f_+(x) = c_3\, e^{p_3\, x}.$$

Next, examine (4.19) in the form

$$g_+''(x) - k^2\, g_+(x) = k^2\, f_+(x) + f_+''(x) = c_3\, (k^2 + p_3^2)\, e^{p_3\, x}$$

with the solution

$$g_+(x) = \beta_1\, e^{+k\, x} + \beta_2\, e^{-k\, x} + c_3\, \frac{k^2 + p_3^2}{p_3^2 - k^2}\, e^{p_3\, x}.$$

Since this solution has to remain bounded as $x \to +\infty$, we find

$$g_+(x) = \beta_2\, e^{-k\, x} + c_3\, \frac{k^2 + p_3^2}{p_3^2 - k^2}\, e^{p_3\, x}.$$

The case $\gamma \geq 0$ and $x \to -\infty$, leads in an analogous way to

$$f(x) \approx f_-(x) = c_1\, e^{p_1\, x} + c_2\, e^{p_2\, x}$$

and

$$g(x) \approx g_-(x) = \beta_1\, e^{+k\, x} + c_1\, \frac{k^2 + p_1^2}{p_1^2 - k^2}\, e^{p_1\, x} + c_2\, \frac{k^2 + p_2^2}{p_2^2 - k^2}\, e^{p_2\, x}.$$

For $0 < k^2 \ll 1$ we find

$$p_1^2 \approx \frac{4^2}{(4 - 3\, k^2)^2}\, \gamma^2 \approx \gamma^2 \tag{4.22}$$

and thus the above formula for $g(x)$ may not be valid for $\gamma^2 \approx k^2 \ll 1$. For $k = 0$ the equation for g simplifies to $g''(x) = f''(x)$, and the conditions at $x = \pm\infty$ imply $g(x) = f(x)$.

4.3.2. *Solutions for intermediate* $|x|$. Choosing a large value for $M > 0$, we construct nontrivial solutions to (4.14)–(4.15) for $-\infty < x \leq -M$, then for $-M \leq x \leq +M$, and then for $+M \leq x < +\infty$. We seek nonzero values of the parameters $(c_1,\, c_2,\, c_3,\, \beta_1,\, \beta_2)$, such that we find a nonzero solution.

Use the solutions f_- and g_- to define a matrix

$$\mathbf{T}_- = \begin{bmatrix} e^{-p_1\, M} & e^{-p_2\, M} & 0 & 0 & 0 \\ p_1\, e^{-p_1\, M} & p_2\, e^{-p_2\, M} & 0 & 0 & 0 \\ p_1^2\, e^{-p_1\, M} & p_2^2\, e^{-p_2\, M} & 0 & 0 & 0 \\ \frac{p_1^2+k^2}{p_1^2-k^2}\, e^{-p_1\, M} & \frac{p_2^2+k^2}{p_2^2-k^2}\, e^{-p_2\, M} & 0 & e^{-k\, M} & 0 \\ \frac{(p_1^2+k^2)\, p_1}{p_1^2-k^2}\, e^{-p_1\, M} & \frac{(p_2^2+k^2)\, p_2}{p_2^2-k^2}\, e^{-p_2\, M} & 0 & k\, e^{-k\, M} & 0 \end{bmatrix}$$

with

$$\mathbf{T}_- \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} f(-M) \\ f'(-M) \\ f''(-M) \\ g(-M) \\ g'(-M) \end{pmatrix}$$

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Use the solutions f_+ and g_+ to define a matrix

$$\mathbf{T}_+ = \begin{bmatrix} 0 & 0 & e^{p_3 M} & 0 & 0 \\ 0 & 0 & p_3\, e^{p_3 M} & 0 & 0 \\ 0 & 0 & p_3^2\, e^{p_3 M} & 0 & 0 \\ 0 & 0 & \frac{p_3^2+k^2}{p_3^2-k^2}\, e^{p_3 M} & 0 & e^{-k M} \\ 0 & 0 & \frac{(p_3^2+k^2)\,p_3}{p_3^2-k^2}\, e^{p_3 M} & 0 & -k\, e^{-k M} \end{bmatrix}$$

with

$$\mathbf{T}_+ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} f(+M) \\ f'(+M) \\ f''(+M) \\ g(+M) \\ g'(+M) \end{pmatrix}$$

Then use an ODE solver to examine the system (4.14)–(4.15) on the interval $[-M, M]$ as an initial value problem. This leads to a matrix $\mathbf{T}$ such that

$$\mathbf{T} \begin{pmatrix} f(-M) \\ f'(-M) \\ f''(-M) \\ g(-M) \\ g'(-M) \end{pmatrix} = \begin{pmatrix} f(+M) \\ f'(+M) \\ f''(+M) \\ g(+M) \\ g'(+M) \end{pmatrix}$$

Now we have two methods to compute the values of the solution at $x = +M$, and the system (4.14)–(4.15) has a nonzero solution if and only if

$$D(k,\gamma) \;=\; \det(\mathbf{M}(k,\gamma)) = \det(\mathbf{T}\cdot\mathbf{T}_- - \mathbf{T}_+) = 0 \tag{4.23}$$

Thus we examine solutions of this equation as function of the parameters k and γ.

4.4. The special case $k = 0$. For $k = 0$, equation (4.15) reads as $g''(x) = f''(x)$ and the only solution satisfying $g(\pm\infty) = f(\pm\infty)$ is $g(x) = f(x)$. Then (4.14) leads to

$$\frac{d}{dx}\begin{pmatrix} f \\ f' \\ f'' \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4\,\gamma + 6\,u_0' & 4 + 6\,u_0 & 0 \end{bmatrix} \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix} \tag{4.24}$$

Using the same approach as in the previous section we define

$$\mathbf{T}_- \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} f(-M) \\ f'(-M) \\ f''(-M) \end{pmatrix} = \begin{bmatrix} e^{-p_1 M} & e^{-p_2 M} & 0 \\ p_1\, e^{-p_1 M} & p_2\, e^{-p_2 M} & 0 \\ p_1^2\, e^{-p_1 M} & p_2^2\, e^{-p_2 M} & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\mathbf{T}_+ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} f(+M) \\ f'(+M) \\ f''(+M) \end{pmatrix} = \begin{bmatrix} 0 & 0 & e^{p_3 M} \\ 0 & 0 & p_3\, e^{p_3 M} \\ 0 & 0 & p_3^2\, e^{p_3 M} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\mathbf{T}_0 \begin{pmatrix} f(-M) \\ f'(-M) \\ f''(-M) \end{pmatrix} = \begin{pmatrix} f(+M) \\ f'(+M) \\ f''(+M) \end{pmatrix}$$

where $\mathbf{T}_0$ is constructed using an ODE solver for (4.24). Then the system (4.14)–(4.15) has a nonzero solution for $k = 0$ if and only if

$$D_0(\gamma) \;=\; \det(\mathbf{M}_0(\gamma)) = \det(\mathbf{T}_0\cdot\mathbf{T}_- - \mathbf{T}_+) = 0 \tag{4.25}$$

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5. Numerical Results on the Instabilities of Plane-Wave Soliton Solutions

5.1. Locating unstable solutions. For our numerical test we used $M = 5$ and for $|x| > 5$ we know $|u_0(x)| = 2\cosh^{-2}(x) \le 3.6 \cdot 10^{-4} \ll 2$ and $|u_0'(x)| \le 7.3 \cdot 10^{-4} \ll 2$. Tests with larger values for M did not change the result significantly. We used the numerical ODE solver `lsode` provided by *Octave*. Based on expression (4.23) we generate plots of the function $D(k, \gamma) = \det(\mathbf{M}(k, \gamma))$ on a domain $0 \le k \le 1$ and $0 \le \gamma \le 0.5$, leading to Figure 2. In the corner $k \approx 1$ and $\gamma \approx 0.5$ the real part of the function vanishes, but a second plot verifies that the imaginary part is different from zero. Thus Figure 2 indicates that we have a clearly defined solution curve of $\det(\mathbf{M}(k, \gamma)) = 0$, away from the origin.

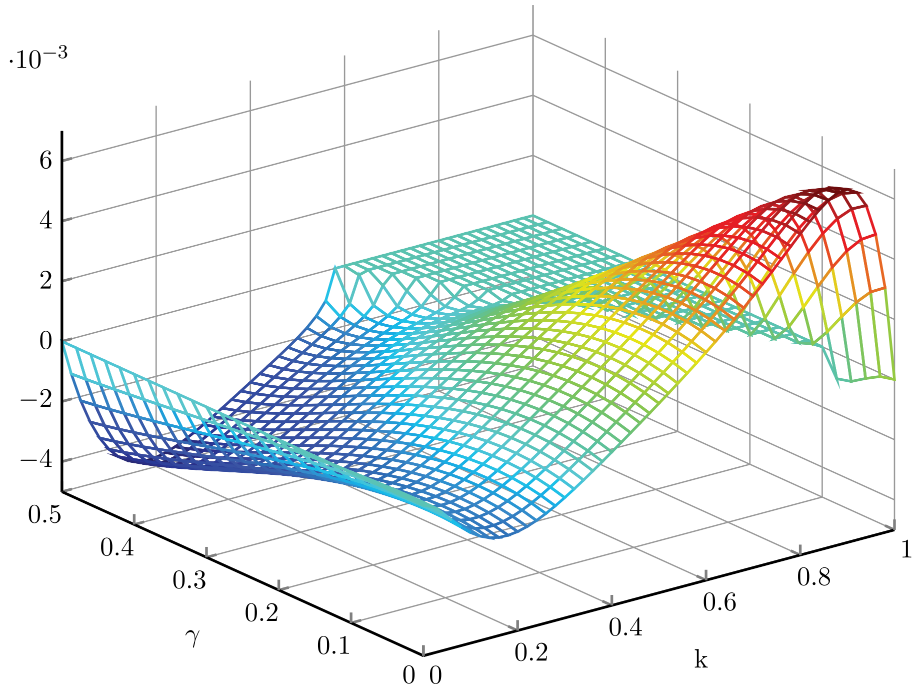

Figure 2. The real part of $\det(\mathbf{M})$ as a function of k and γ

Since the behavior close to $(k, \gamma) \approx (0, 0)$ is critical, we examine this section with a finer resolution, leading to Figure 3(a). The obvious spikes are caused by the zeros in the denominator in condition (4.22). Figure 3(a) suggests the existence of a solution along the axis $k = 0$. Using (4.25) we generate Figure 3(b). As a consequence the only solution along the axis $k = 0$ is at $\gamma = 0$. Thus the known solution (4.16) is an isolated solution in the parameter space (k, γ) at $(0, 0)$.

With the above preparation we can now construct values of (k, γ) leading to nonzero solutions of (4.14)–(4.15) and thus for $\gamma > 0$ to unstable soliton solutions of the NV equations (1.2)–(1.4).

We trace a solution curve of (4.23) by generating an arc length parametrization of the curve. Using an arbitrary initial point on the curve (use a contour plot of

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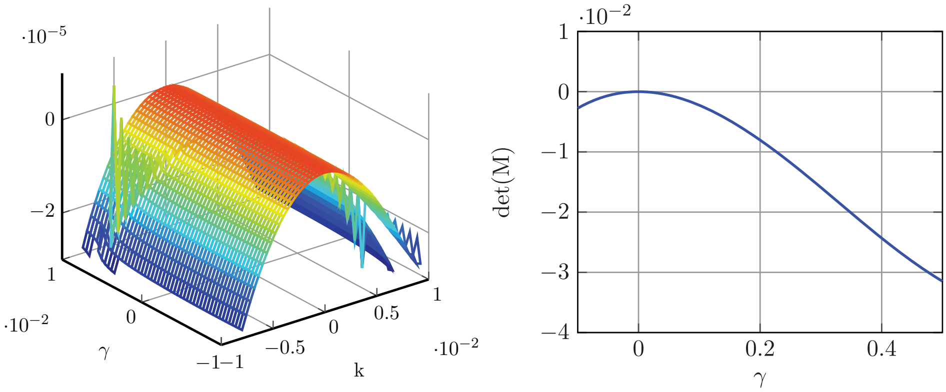
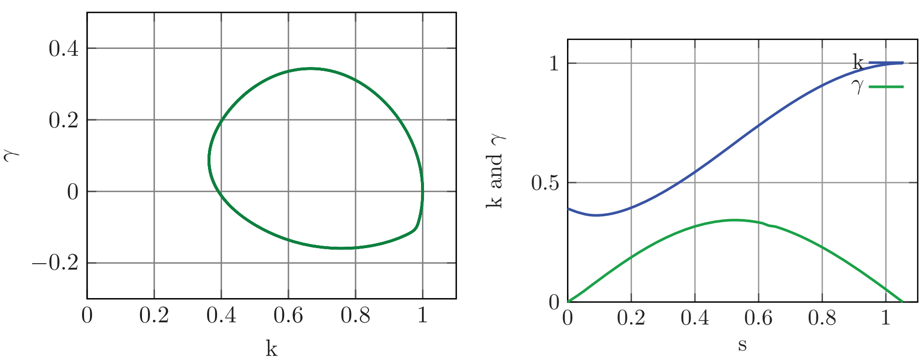


(a) det $\mathbf{M}(k,\gamma)$ close to origin (b) det $\mathbf{M}_0(\gamma)$

FIGURE 3. Behavior at the origin and along the axis $k = 0$

Figure 2) and a starting direction, we minimize $|\det(\mathbf{M}(k,\gamma))|$ along a straight line segment orthogonal to the stepping direction. Using this local minimum, we adjust the stepping direction and then make a small step to follow the solution curve. While stepping along the curve we verify that we actually have a solution of (4.23), and not only a minimum. This proved to be a stable algorithm and generated Figure 4.

Observe that Figure 4(a) also displays negative values of γ, which do not lead to unstable solutions of (1.2)–(1.4). We display these values to confirm that we have a closed curve without branching points.



(a) solution curve of $|\det(\mathbf{M})| = 0$ (b) parameters leading to unstable solutions

FIGURE 4. Solution curve of $|\det(\mathbf{M})| = 0$ and parameters k and γ leading to unstable solutions

For the parameter values of (k,γ) in Figure 4(b) there are nonzero functions $f(x)$, $g(x)$ and $h(x)$, such that for small ϵ we have solutions of the equations (4.2)–(4.4) of the form (4.5)–(4.7). Since the y dependence of these functions is of the form e^{iky}, the functions are periodic in y with a period of $L_y = \frac{2\pi}{k}$. The corresponding

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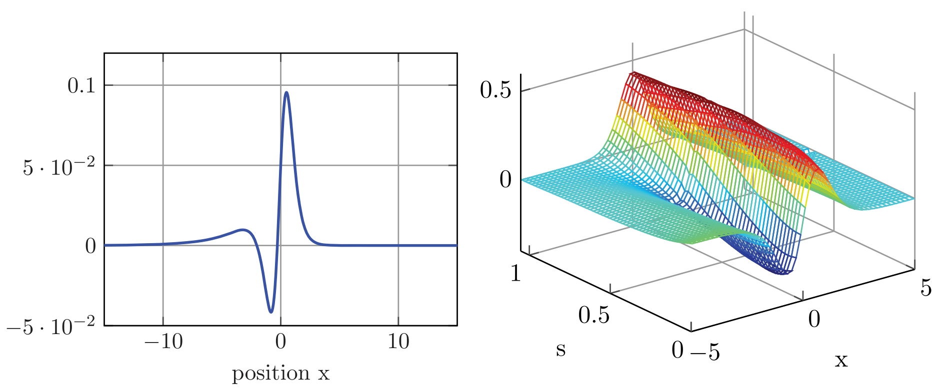
exponent is shown in Figure (4(b)). Thus the Novikov–Veselov equations (1.2)–(1.4) with initial condition $u(x, y, 0) = -2\,c\,\mathrm{sech}^2(\sqrt{c}\,x)$ will lead to an unstable soliton solution.

THEOREM 3. *The soliton solutions* $u(x, y, t) = v(x, y, t) = -2\,c\,\mathrm{sech}^2(\sqrt{c}\,(x - c\,t))$ *and* $w(x, y, t) = 0$ *of the NV equations* (1.2)–(1.4) *are not stable. For values of* $0.363 < k \leq 1$ *there are y–periodic, unstable contributions with a period of* $L_y = \frac{2\,\pi}{k\,\sqrt{c}}$.

5.2. Constructing unstable solutions numerically. As an example, in this section we construct one of the above unstable solutions numerically, using the algorithm from Section 3 for periodic solutions in x and y. Since the soliton solution $\mathrm{sech}^2(x)$ decays rapidly, the above results still apply when working on a sufficiently large domain. The steps of the algorithm are as follows:

(1) Choose values of (k, γ) in Figure 4(b) and determine the eigenvector $(c_1, c_2, c_3, \beta_1, \beta_2)$ for the zero eigenvalue.
(2) Use the algorithm leading to the matrices $\mathbf{T}$, $\mathbf{T}_-$ and $\mathbf{T}_+$ to construct the nonzero functions $f(x), g(x), h(x)$.
(3) Pick a size domain such that $\frac{2\pi}{k}$ periodic functions in y are admissable.
(4) Construct initial values, using (4.8)–(4.10), use a small value of ϵ. Without taking the transformations (4.1) into account we obtain a speed of $c = 1$ of the unperturbed soliton.
(5) Solve the NV equations (1.2)–(1.4), using the algorithm presented in Section 3.
(6) The deviation from the single soliton solution should not change its shape, but the size is expected to be proportional to $e^{\gamma\,t}$.
(7) Solutions for speeds $c \neq 1$ can be constructed similarly, using the transformations (4.1).



(a) Cross-section of the initial perturbation with $k \approx 0.5$ and $\gamma \approx 0.3$

(b) Perturbation as a function of position x and arc length s

FIGURE 5. The shape of the perturbation function $f(x)$ from equation (4.5)

The evolution of a perturbed soliton from Theorem 3 was computed using the semi-implicit pseudo-spectral method. As initial data, we chose a perturbed KdV

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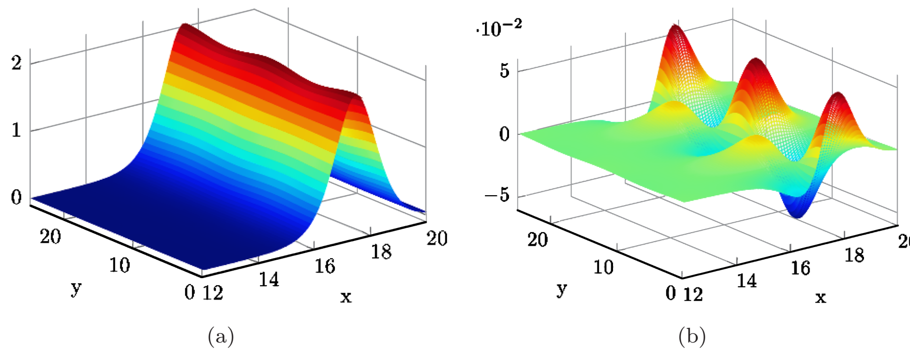

FIGURE 6. a) the perturbed solution $-u(x, y, t)$ at $t = 5$. b) difference from the unperturbed soliton at time $t = 5$.

soliton with speed $c = 1$, starting at $x = 10$. The perturbation had an initial amplitude of 0.0095. For the domain size we chose $W_x = W_y = \frac{4\pi}{k\,c} \approx 25$. The shape of the perturbation $f(x)$ is found in Figure 5(a), and the values $k = 0.504$ and $\gamma = 0.296$ were chosen in Figure 4(b). The values of the exact KdV soliton at the boundaries $x = 0$ and $x = W_x$ are smaller than $2 \cdot 10^{-8}$, and we apply periodic boundary conditions. A time-snapshot of the solution at time $t = 5$ and the difference to the unperturbed KdV soliton at time $t = 5$ are found in Figure 6. Note that in Figure 6 $-u$ is plotted for easier viewing. In Figure 6(a) the deviation from the KdV soliton $-2\,\cosh^{-2}(x - t - 10)$ is barely visible, but Figure 6(b) shows the difference. The L_2 norm of the difference of the computed solution and the known exact KdV soliton follows the expected exponential curve $e^{\gamma t}$ very closely. The front edge in Figure 6(b) matches the shape of the function $f(x)$ in Figure 5(a). Animations of the time evolution of the perturbed solution are available on the web site [**30**]. Similar computations with initially smaller perturbations confirm the above and thus the instability result in Theorem 3.

One can construct the shapes of the functions $f(x)$ for all positive values of γ along the arc in Figure 4 to obtain Figure 5(b). The solutions constructed for $(k, \gamma) = (1, 0)$ must match the known exact solutions (4.17), which is confirmed.

6. Conclusions

In this work a semi-implicit pseudo-spectral method was introduced for the numerical computation of evolutions of solutions to the NV equation and used to compute the evolution of an unstable soliton solution. A linear stability analysis yields a stability condition for the Crank-Nicolson scheme in the pseudo-spectral method on the linearized IBVP, which is also valid for the NV equation with nonzero energy. The instability of traveling wave solutions to transverse perturbations was established by the **K**-expansion method, and unstable soliton solutions were constructed.

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