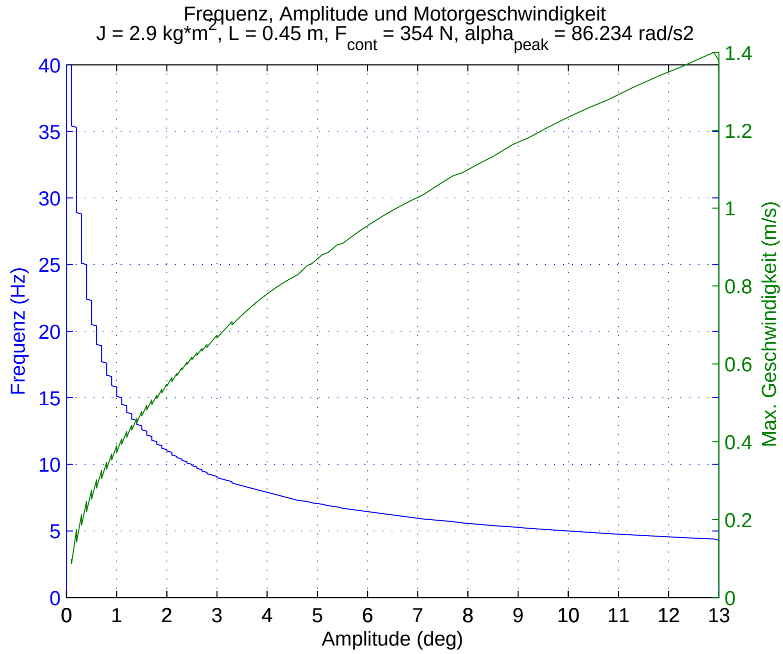

Figure 5.12: System limits in frequency and amplitude for maximum continuous loading

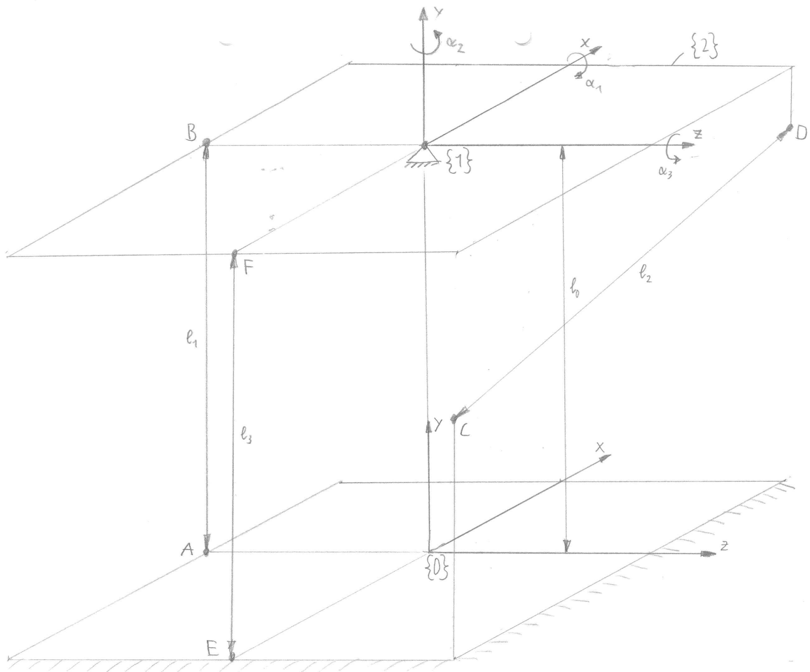

Figure 5.13: Schematic representation of the *SpineMime* kinematics

Motor length and platform angle vectors

$$\vec{L} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Points B, D and F expressed in $\{1\}$, motion base in zero position

$$^1\vec{r}_{\mathrm{B}_0} = \begin{pmatrix} x_{\mathrm{B}_0} \\ y_{\mathrm{B}_0} \\ z_{\mathrm{B}_0} \end{pmatrix} \quad (5.11) \qquad ^1\vec{r}_{\mathrm{D}_0} = \begin{pmatrix} x_{\mathrm{D}_0} \\ y_{\mathrm{D}_0} \\ z_{\mathrm{D}_0} \end{pmatrix} \quad (5.12) \qquad ^1\vec{r}_{\mathrm{F}_0} = \begin{pmatrix} x_{\mathrm{F}_0} \\ y_{\mathrm{F}_0} \\ z_{\mathrm{F}_0} \end{pmatrix} \quad (5.13)$$

Motion of the platform = rotation of the position vectors

Rotation around x-axis with rotation matrix R_x and angle α_1

$$^1\vec{r}_{\mathrm{B}} = R_x(\alpha_1)\,^1\vec{r}_{\mathrm{B}_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_1) & -\sin(\alpha_1) \\ 0 & \sin(\alpha_1) & \cos(\alpha_1) \end{bmatrix} \begin{pmatrix} x_{\mathrm{B}_0} \\ y_{\mathrm{B}_0} \\ z_{\mathrm{B}_0} \end{pmatrix} \tag{5.14}$$

$$^1\vec{r}_{\mathrm{D}} = R_x(\alpha_1)\,^1\vec{r}_{\mathrm{D}_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_1) & -\sin(\alpha_1) \\ 0 & \sin(\alpha_1) & \cos(\alpha_1) \end{bmatrix} \begin{pmatrix} x_{\mathrm{D}_0} \\ y_{\mathrm{D}_0} \\ z_{\mathrm{D}_0} \end{pmatrix} \tag{5.15}$$

$$^1\vec{r}_{\mathrm{F}} = R_x(\alpha_1)\,^1\vec{r}_{\mathrm{F}_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_1) & -\sin(\alpha_1) \\ 0 & \sin(\alpha_1) & \cos(\alpha_1) \end{bmatrix} \begin{pmatrix} x_{\mathrm{F}_0} \\ y_{\mathrm{F}_0} \\ z_{\mathrm{F}_0} \end{pmatrix} \tag{5.16}$$

Rotation around y-axis with rotation matrix R_y and angle α_2

$$^1\vec{r}_{\mathrm{B}} = R_y(\alpha_2)\,^1\vec{r}_{\mathrm{B}_0} = \begin{bmatrix} \cos(\alpha_2) & 0 & \sin(\alpha_2) \\ 0 & 1 & 0 \\ -\sin(\alpha_2) & 0 & \cos(\alpha_2) \end{bmatrix} \begin{pmatrix} x_{\mathrm{B}_0} \\ y_{\mathrm{B}_0} \\ z_{\mathrm{B}_0} \end{pmatrix} \tag{5.17}$$

$$^1\vec{r}_{\mathrm{D}} = R_y(\alpha_2)\,^1\vec{r}_{\mathrm{D}_0} = \begin{bmatrix} \cos(\alpha_2) & 0 & \sin(\alpha_2) \\ 0 & 1 & 0 \\ -\sin(\alpha_2) & 0 & \cos(\alpha_2) \end{bmatrix} \begin{pmatrix} x_{\mathrm{D}_0} \\ y_{\mathrm{D}_0} \\ z_{\mathrm{D}_0} \end{pmatrix} \tag{5.18}$$

$$^1\vec{r}_{\mathrm{F}} = R_y(\alpha_2)\,^1\vec{r}_{\mathrm{F}_0} = \begin{bmatrix} \cos(\alpha_2) & 0 & \sin(\alpha_2) \\ 0 & 1 & 0 \\ -\sin(\alpha_2) & 0 & \cos(\alpha_2) \end{bmatrix} \begin{pmatrix} x_{\mathrm{F}_0} \\ y_{\mathrm{F}_0} \\ z_{\mathrm{F}_0} \end{pmatrix} \tag{5.19}$$

Rotation around z-axis with rotation matrix R_z and angle α_3

$$^1\vec{r}_{\mathrm{B}} = R_z(\alpha_3)\,^1\vec{r}_{\mathrm{B}_0} = \begin{bmatrix} \cos(\alpha_3) & -\sin(\alpha_3) & 0 \\ \sin(\alpha_3) & \cos(\alpha_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_{\mathrm{B}_0} \\ y_{\mathrm{B}_0} \\ z_{\mathrm{B}_0} \end{pmatrix} \tag{5.20}$$

$$^1\vec{r}_{\mathrm{D}} = R_z(\alpha_3)\,^1\vec{r}_{\mathrm{D}_0} = \begin{bmatrix} \cos(\alpha_3) & -\sin(\alpha_3) & 0 \\ \sin(\alpha_3) & \cos(\alpha_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_{\mathrm{D}_0} \\ y_{\mathrm{D}_0} \\ z_{\mathrm{D}_0} \end{pmatrix} \tag{5.21}$$

$$^1\vec{r}_{\mathrm{F}} = R_z(\alpha_3)\,^1\vec{r}_{\mathrm{F}_0} = \begin{bmatrix} \cos(\alpha_3) & -\sin(\alpha_3) & 0 \\ \sin(\alpha_3) & \cos(\alpha_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_{\mathrm{F}_0} \\ y_{\mathrm{F}_0} \\ z_{\mathrm{F}_0} \end{pmatrix} \tag{5.22}$$

The three single rotations around the x-, y- and z-axis can be done in one matrix operation with the rotation matrix:

$$M_{zyx}(\alpha_3, \alpha_2, \alpha_1) = R_z(\alpha_3) R_y(\alpha_2) R_x(\alpha_1) \tag{5.23}$$

We get $M_{zyx}(\alpha_3, \alpha_2, \alpha_1)$ as (sin = s, cos = c):

$$M_{zyx}(\alpha_3, \alpha_2, \alpha_1) = \begin{bmatrix} \mathrm{c}\alpha_2\mathrm{c}\alpha_3 & \mathrm{c}\alpha_3\mathrm{s}\alpha_1\mathrm{s}\alpha_2 - \mathrm{c}\alpha_1\mathrm{s}\alpha_3 & \mathrm{c}\alpha_1\mathrm{s}\alpha_3\mathrm{s}\alpha_2 + \mathrm{s}\alpha_1\mathrm{s}\alpha_3 \\ \mathrm{c}\alpha_2\mathrm{s}\alpha_3 & \mathrm{s}\alpha_1\mathrm{s}\alpha_2\mathrm{s}\alpha_3 + \mathrm{c}\alpha_1\mathrm{c}\alpha_3 & \mathrm{c}\alpha_1\mathrm{s}\alpha_2\mathrm{s}\alpha_3 - \mathrm{c}\alpha_3\mathrm{s}\alpha_1 \\ -\mathrm{s}\alpha_2 & \mathrm{c}\alpha_2\mathrm{s}\alpha_1 & \mathrm{c}\alpha_1\mathrm{c}\alpha_2 \end{bmatrix} \tag{5.24}$$

In this we way we have for the position vectors:

$$^1\vec{r}_{\mathrm{B}} = M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\,^1\vec{r}_{\mathrm{B}_0} \tag{5.25}$$

$$^1\vec{r}_{\mathrm{D}} = M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\,^1\vec{r}_{\mathrm{D}_0} \tag{5.26}$$

$$^1\vec{r}_{\mathrm{F}} = M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\,^1\vec{r}_{\mathrm{F}_0} \tag{5.27}$$

Frame transformation

We want to express the position vectors in reference frame $\{0\}$. For this we have to express the origin of frame $\{1\}$ in frame $\{0\}$. This is done with the transformation vector which is:

$$^0\vec{r}_{1,\mathrm{org}} = \begin{pmatrix} 0 \\ l_0 \\ 0 \end{pmatrix} \tag{5.28}$$

The position vectors expressed in frame $\{0\}$ are just an addition of the position vector expressed in frame $\{1\}$ with the transformation vector:

$$^0\vec{r}_{\mathrm{B}} = {}^0\vec{r}_{1,\mathrm{org}} + {}^1\vec{r}_{\mathrm{B}} \tag{5.29}$$

$$^0\vec{r}_{\mathrm{D}} = {}^0\vec{r}_{1,\mathrm{org}} + {}^1\vec{r}_{\mathrm{D}} \tag{5.30}$$

$$^0\vec{r}_{\mathrm{F}} = {}^0\vec{r}_{1,\mathrm{org}} + {}^1\vec{r}_{\mathrm{B}} \tag{5.31}$$

Points A, C and E expressed in $\{0\}$

$$^0\vec{r}_{\mathrm{A}} = \begin{pmatrix} x_{\mathrm{A}} \\ y_{\mathrm{A}} \\ z_{\mathrm{A}} \end{pmatrix} \tag{5.32}$$

$$^0\vec{r}_{\mathrm{C}} = \begin{pmatrix} x_{\mathrm{C}} \\ y_{\mathrm{C}} \\ z_{\mathrm{C}} \end{pmatrix} \tag{5.33}$$

$$^0\vec{r}_{\mathrm{E}} = \begin{pmatrix} x_{\mathrm{E}} \\ y_{\mathrm{E}} \\ z_{\mathrm{E}} \end{pmatrix} \tag{5.34}$$

Motor lengths

The motor lengths correspond to the norms of the vectors $\overrightarrow{\mathrm{AB}}$, $\overrightarrow{\mathrm{CD}}$ and $\overrightarrow{\mathrm{EF}}$. We have:

$$\overrightarrow{\mathrm{AB}} = {}^0\vec{r}_{\mathrm{B}} - {}^0\vec{r}_{\mathrm{A}} \quad \Rightarrow \quad l_1 = |\overrightarrow{\mathrm{AB}}| \tag{5.35}$$

$$\overrightarrow{\mathrm{CD}} = {}^0\vec{r}_{\mathrm{D}} - {}^0\vec{r}_{\mathrm{C}} \quad \Rightarrow \quad l_2 = |\overrightarrow{\mathrm{CD}}| \tag{5.36}$$

$$\overrightarrow{\mathrm{AB}} = {}^0\vec{r}_{\mathrm{F}} - {}^0\vec{r}_{\mathrm{E}} \quad \Rightarrow \quad l_3 = |\overrightarrow{\mathrm{EF}}| \tag{5.37}$$

Substituting with above derivations lead to:

$$l_1 = |{}^0\vec{r}_{1,\mathrm{org}} + M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\, {}^1\vec{r}_{\mathrm{B}_0} - {}^0\vec{r}_{\mathrm{A}}| \tag{5.38}$$

$$l_2 = |{}^0\vec{r}_{1,\mathrm{org}} + M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\, {}^1\vec{r}_{\mathrm{D}_0} - {}^0\vec{r}_{\mathrm{C}}| \tag{5.39}$$

$$l_3 = |{}^0\vec{r}_{1,\mathrm{org}} + M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\, {}^1\vec{r}_{\mathrm{F}_0} - {}^0\vec{r}_{\mathrm{E}}| \tag{5.40}$$

Finally we get the motor length vector:

$$\vec{L} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} |{}^0\vec{r}_{1,\mathrm{org}} + M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\, {}^1\vec{r}_{\mathrm{B}_0} - {}^0\vec{r}_{\mathrm{A}}| \\ |{}^0\vec{r}_{1,\mathrm{org}} + M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\, {}^1\vec{r}_{\mathrm{D}_0} - {}^0\vec{r}_{\mathrm{C}}| \\ |{}^0\vec{r}_{1,\mathrm{org}} + M_{zyx}(\alpha_3, \alpha_2, \alpha_1)\, {}^1\vec{r}_{\mathrm{F}_0} - {}^0\vec{r}_{\mathrm{E}}| \end{pmatrix} \tag{5.41}$$

Approximation for small angles

Calculating the motor lengths $\vec{L}$ after equation 5.41 is a nonlinear problem due to the sine and cosine operations in the rotation matrix $M_{zyx}(\alpha_3, \alpha_2, \alpha_1)$. Concerning the computational effort, the problem was discussed with Prof. Dr. Andreas Stahel, professor for mathematics at the Bern University of Applied Sciences. It is to consider that the motor lengths have to calculated inside an adequate time period for guaranteeing real-time operation of the robotic system. The change of the motor lengths can be approximated through linearization of the problem. The inverse kinematics can be approximated with:

$$\boxed{\Delta\vec{L} = A\Delta\vec{\alpha}} \tag{5.42}$$

The forward kinematics can be approximated with:

$$\boxed{\Delta\vec{\alpha} = A^{-1}\Delta\vec{L}} \tag{5.43}$$

The approximation matrix A relates $\vec{L}$ and $\vec{\alpha}$ linearly. It is built from the partial derivations of $\vec{L}$ after $\vec{\alpha}$ for $\alpha = 0$. The equations will be approximatively valid for small variations from the zero position of the platform. The approximation matrix is calculated as:

$$A = \left.\frac{\partial\vec{L}}{\partial\vec{\alpha}}\right|_{\vec{\alpha}=0} = \begin{bmatrix} \frac{\partial l_1}{\partial\alpha_1} & \frac{\partial l_1}{\partial\alpha_2} & \frac{\partial l_1}{\partial\alpha_3} \\ \frac{\partial l_2}{\partial\alpha_1} & \frac{\partial l_2}{\partial\alpha_2} & \frac{\partial l_2}{\partial\alpha_3} \\ \frac{\partial l_3}{\partial\alpha_1} & \frac{\partial l_3}{\partial\alpha_2} & \frac{\partial l_3}{\partial\alpha_3} \end{bmatrix} \tag{5.44}$$

Numerical system description

The system is now described with its numerical values. The kinematics calculation scripts `SpineMime_EulerRot.m` and `SpineMime_AxisAngleRot.m` are available in work package folder `AP313_Kinematik_Aufbau`.

Motion base The motion base coordinates in relation to the motion base frame {1} are:

$$ {}^{1}\vec{r}_{\mathrm{B}_0} = \begin{pmatrix} 0 \\ 0.04 \\ -0.45 \end{pmatrix} \mathrm{m} \qquad {}^{1}\vec{r}_{\mathrm{D}_0} = \begin{pmatrix} 0.3 \\ -0.04 \\ 0.45 \end{pmatrix} \mathrm{m} \qquad {}^{1}\vec{r}_{\mathrm{F}_0} = \begin{pmatrix} -0.45 \\ 0.04 \\ 0 \end{pmatrix} \mathrm{m} $$

Frame transformation Frame {1} expressed in frame {0} is:

$$ {}^{0}\vec{r}_{1,\mathrm{org}} = \begin{pmatrix} 0 \\ 0.668 \\ 0 \end{pmatrix} \mathrm{m} $$

Ground The ground coordinates expressed in the ground frame {0} are:

$$ {}^{0}\vec{r}_{\mathrm{A}} = \begin{pmatrix} 0 \\ 0 \\ -0.45 \end{pmatrix} \mathrm{m} \qquad {}^{0}\vec{r}_{\mathrm{C}} = \begin{pmatrix} -0.4 \\ 0.473 \\ 0.45 \end{pmatrix} \mathrm{m} \qquad {}^{0}\vec{r}_{\mathrm{F}} = \begin{pmatrix} -0.45 \\ 0 \\ 0 \end{pmatrix} \mathrm{m} $$

Approximation matrix With the above values the approximation matrix for the inverse kinematics is calculated to:

$$ A = \begin{bmatrix} 0.4500 & 0 & 0 \\ -0.0973 & 0.4394 & 0.1039 \\ 0 & 0 & -0.4500 \end{bmatrix} \frac{\mathrm{m}}{\mathrm{rad}} $$

Approximation errors The motor lengths were calculated for different platform angles and different rotation directions. Platform angles up to the maximum range of motion (13°) were respected. The values were one time calculated exact and one time with the approximation strategy. The results were compared and the approximation errors were analyzed. The absolute motor length errors are shown in figure 5.14. It is to see that the errors are growing exponentially with increasing platform angle. In the graph for the y-axis rotations only two data lines can be seen. This is because the errors for length 1 and length 3 are exactly the same and the lines are superimposed. The absolute errors are always below 8 mm. 8 mm corresponds approximately to one degree. We can say that the platform angle errors will be below one degree as long as it is only rotated about the x-, y- or z-axis. If two or three axes are involved in the platform displacement the errors would add up. Even in this situation the maximum positioning error can be estimated to be below two degrees. The actual application doesn't require high precision, because no positioning tasks have to be done. The main focus is to generate vibrations for biomechanical stimulation. Precision is absolutely not of interest when stochastic motion is desired. From this point of view it can be said that the approximation strategy is adequate. Further it reduces the computational costs.

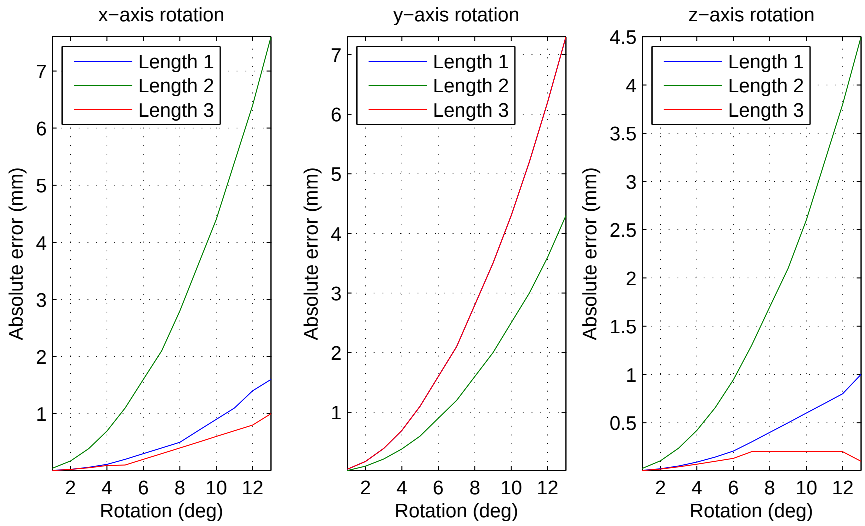

Figure 5.14: Approximation errors for different rotation directions

5.2.3 Axis-Angle rotation

In the description above, the platform position was described using the Euler angles α_1, α_2 and α_3. Now we consider the definition of a desired motion for stimulation. This could for example be a sine of the form $\varphi(t) = \phi \sin(2\pi f t)$ (see also section 4.4). This would work fine if the platform rotates around the x-, y- or z-axis when ϕ corresponds to either α_1, α_2 or α_3. If we want to have a rotation axis different from the x-, y- or z-axis the amplitude ϕ won't correspond to either α_1, α_2 or α_3. Such a motion can be described using the axis-angle form. A rotation axis $\vec{k}$ with $|\vec{k}| = 1$ and a rotation angle ϕ are needed. A position vector can be rotated about an arbitrary axis $\vec{k}$ together with the general rotation matrix $R_{\vec{k}}$ [48] (cos = c, sin = s, versine = 1-cos = v).

$$\vec{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \qquad |\vec{k}| = 1 \tag{5.45}$$

$$R_{\vec{k}} = \begin{bmatrix} k_x^2 \mathrm{v}\phi + \mathrm{c}\phi & k_x k_y \mathrm{v}\phi - k_z \mathrm{s}\phi & k_y k_z \mathrm{v}\phi + k_y \mathrm{s}\phi \\ k_x k_y \mathrm{v}\phi + k_z \mathrm{s}\phi & k_y^2 \mathrm{v}\phi + \mathrm{c}\phi & k_y k_z \mathrm{v}\phi - k_x \mathrm{s}\phi \\ k_x k_z \mathrm{v}\phi - k_y \mathrm{s}\phi & k_y k_z \mathrm{v}\phi + k_x \mathrm{s}\phi & k_z^2 \mathrm{v}\phi + \mathrm{c}\phi \end{bmatrix} \tag{5.46}$$

For performing the approximated inverse kinematics after equation 5.42 we need to know the Euler angles α_1, α_2 and α_3. Having numerical values for $\vec{k}$ and ϕ we know the rotation matrix. The same rotation matrix would result the values for α_1, α_2 and α_3 are set into $M_{zyx}(\alpha_3, \alpha_2, \alpha_1)$ (see equation 5.24). The other way around α_1, α_2 and α_3 can be calculated out of the known rotation matrix (for derivation see [48]). We get the Euler angles with the following formulas. Therein r_{ij} represents the element at the i-th row and j-th column from the rotation matrix.