# Biot–Savart and Helmholtz Coils

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## Contents



# 1 The law of Biot–Savart

If a short segment $\vec{ds}$ of a conductor carries a current $I$ then the contribution $d\vec{H}$ to the magnetic field is given by the law of Biot–Savart

$$d\vec{H} = \frac{I}{4\,\pi\,r^3}\,\vec{ds}\times\vec{r} \tag{1}$$

where $\vec{r}$ is the vector connecting the point on the conductor to the points at which the field is to be computed. The situation is visualized by Figure 1.

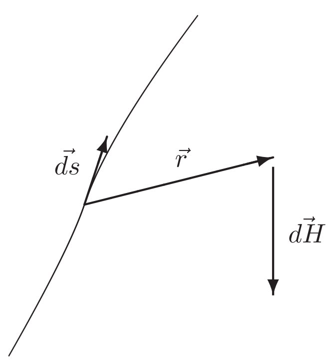


Figure 1: Law of Biot–Savart for magnetic fields

The shape of a current carrying wire can be described by a curve $C$. The resulting magnetic field $\vec{H}$ can be computed by an integral

$$\vec{H} = \int_C d\vec{H} = \frac{I}{4\,\pi}\int_C \frac{1}{r^3}\,\vec{ds}\times\vec{r}$$

We will examine a few important configurations.

## 2 Magnetic field of a straight line wire

A wire is placed along the $z$–axis in $\mathbb{R}^3$ and carries a current $I$. The resulting magnetic field $\vec{H}$ will not depend on $z$ and the strength of the field will depend only on the distance from the wire ($z$–axis). Thus we compute the field only along the $x$–axis.

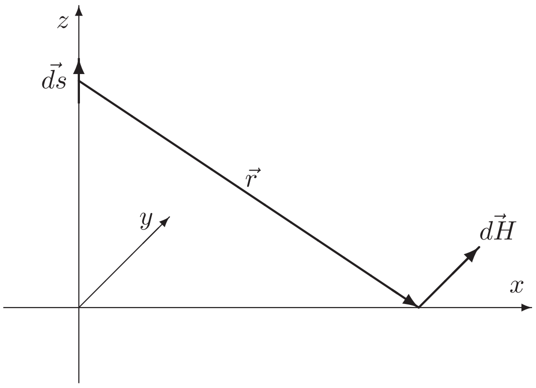


Figure 2: Magnetic field by a straight line wire

Examine a section of length $\Delta z$ at height $z$ on the $z$–axis. According the the law of Biot–Savart (1) and Figure 2 we obtain

$$\vec{ds} = \begin{pmatrix} 0 \\ 0 \\ \Delta z \end{pmatrix} \quad \text{and} \quad \vec{r} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ -z \end{pmatrix}$$

and thus a resulting contribution the the magnetic field

$$\overrightarrow{\Delta H} \approx \frac{I}{4\,\pi\,r^3} \begin{pmatrix} 0 \\ 0 \\ \Delta z \end{pmatrix} \times \begin{pmatrix} x \\ 0 \\ -z \end{pmatrix} = \frac{I}{4\,\pi\,r^3} \begin{pmatrix} 0 \\ x\,\Delta z \\ 0 \end{pmatrix}$$

where $r = \sqrt{x^2 + z^2}$. We observe that only the $y$–component is different from zero and we obtain

$$\Delta H_y = \frac{I\,x}{4\,\pi\,\sqrt{x^2 + z^2}^3}\,\Delta z$$

To compute the total magnetic field we have to integrate the above over the domain $-\infty < z < \infty$ and arrive at

$$H_y = \frac{x\,I}{4\,\pi} \int_{-\infty}^{\infty} \frac{1}{(x^2 + z^2)^{3/2}}\,dz = \frac{x\,I}{4\,\pi} \frac{z}{x^2\,\sqrt{x^2 + z^2}} \Big|_{z=-\infty}^{\infty} = \frac{I}{2\,\pi\,x}$$

To compute the above improper integral we used the limit

$$\lim_{z \to \infty} \frac{z}{\sqrt{x^2 + z^2}} = 1$$

As a final result we find the magnetic field $h$ along the $x$-axis is pointing in $y$–direction and has strength

$$H = \frac{I}{2\,\pi\,x}$$

This can be confirmed experimentally.

## 3 Magnetic field of a circular conductor

The configuration of a circular conductor has many useful applications. We examine a circle of radius $R$ in the $xy$– plane with center at the origin, as shown in Figure 3.

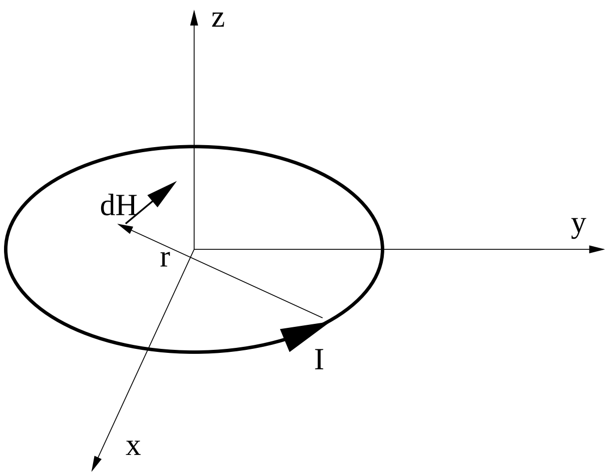


Figure 3: Configuration of a circular conductor

A parametrization of the circle is given by

$$\begin{pmatrix} R\cos\phi \\ R\sin\phi \\ 0 \end{pmatrix} \quad \text{where} \quad 0 \le \phi \le 2\,\pi$$

leading to a line segment $\vec{ds}$ of

$$\vec{ds} = \begin{pmatrix} -R\sin\phi \\ R\cos\phi \\ 0 \end{pmatrix} d\phi$$

The field $\vec{H}$ generated will have a radial symmetry and we may examine the field in the $xz$–plane only, i.e. at points $(x, 0, z)$. We find

$$\vec{r} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} - \begin{pmatrix} R\cos\phi \\ R\sin\phi \\ 0 \end{pmatrix}$$

and

$$\begin{aligned} r^2 &= (R\cos\phi - x)^2 + R^2\sin^2\phi + z^2 \\ &= R^2 - 2\,x\,R\cos\phi + x^2 + z^2 \end{aligned}$$

To apply Biot–Savart we need the expression

$$\vec{ds} \times \vec{r} = \begin{pmatrix} -R\sin\phi \\ R\cos\phi \\ 0 \end{pmatrix} \times \begin{pmatrix} x - R\cos\phi \\ -R\sin\phi \\ z \end{pmatrix} d\phi = \begin{pmatrix} z\,R\cos\phi \\ z\,R\sin\phi \\ R^2 - x\,R\cos\phi \end{pmatrix} d\phi$$

and thus obtain an integral for the 3 components of the field $\vec{H}$ at the point $(x, 0, z)$

$$\vec{H} = \frac{I}{4\,\pi} \int_0^{2\pi} \frac{1}{(R^2 - 2\,x\,R\cos\phi + x^2 + z^2)^{3/2}} \begin{pmatrix} z\,R\cos\phi \\ z\,R\sin\phi \\ R^2 - x\,R\cos\phi \end{pmatrix} d\phi \qquad (2)$$

For each of the three components of the field we find one integral to be computed. We will compute the expression for a few important configurations.

### 3.1 Field along the central axis

Along the $z$–axis we use $x = 0$ and the above integral simplifies to

$$\vec{H} = \frac{I}{4\pi} \int_0^{2\pi} \frac{1}{(R^2+z^2)^{3/2}} \begin{pmatrix} z\,R\,\cos\phi \\ z\,R\,\sin\phi \\ R^2 \end{pmatrix} d\phi$$

Verify that the $x$- and $y$ component of $\vec{H}$ vanish, as they should because of the radial symmetry. For the $z$ component $H_z(z)$ we obtain

$$H_z(z) = \frac{I}{4\pi} \int_0^{2\pi} \frac{R^2}{(R^2+z^2)^{3/2}}\, d\phi = \frac{I}{2}\, \frac{R^2}{(R^2+z^2)^{3/2}}$$

For very small and large values of $z$ the above may be simplified to

$$H_{z\ll R} \approx \frac{I}{2}\,\frac{1}{R} \quad \text{and} \quad H_{z\gg R} \approx \frac{I}{2}\,\frac{R^2}{z^3}$$

The above approximations allow to compute the field at the center of the coil and show that the field along the center axis converges to 0 like $1/z^3$ .

### 3.2 The Helmholtz configuration

For many applications is is important that the magnetic field should be constant over the domain to be examined. The above computations show that the field generated by a single coil is far from constant. For a Helmholtz configuration we place two of the above coils, at the heights $z = \pm h$. The value of $h$ has to be such that the field around the center is as homogeneous as possible. To examine this situation we shift one coil up by $h$ and another coil down by $-h$ and then examine the resulting field. On the left in Figure 4 we find the results if the two coils are close together, on the right if they are far apart. Neither situation is optimal for a homogeneous magnetic field.

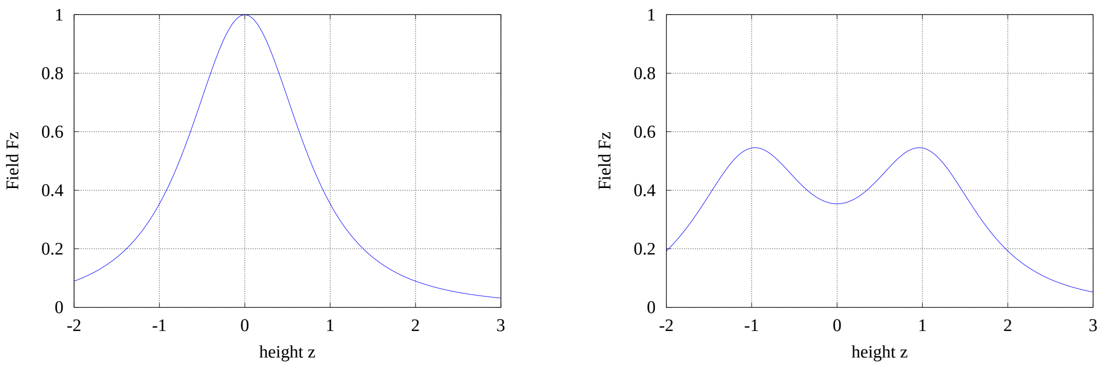


Figure 4: Magnetic field along the central axis

We have to examine the field $G$ generated by both coils and thus $G$ is given as the sum of the two fields by the individual coils at height $z = \pm h$.

$$G(z) = H_z(z+h) + H_z(z-h) = \frac{I}{2}\left(\frac{R^2}{(R^2+(z-h)^2)^{3/2}} + \frac{R^2}{(R^2+(z+h)^2)^{3/2}}\right)$$

The field at $z = 0$ is as homogeneous as possible if as many terms as possible in the Taylor expansion vanish.

$$G(z) \approx G(0) + \frac{d\,G(0)}{dz}\,z + \frac{1}{2}\,\frac{d^2\,G(0)}{dz^2}\,z^2 + \frac{1}{6}\,\frac{d^3\,G(0)}{dz^3}\,z^3 + \ldots$$

Since $H_z$ is an even function know that the first derivative is an odd function and the second derivative is an even function. Thus we find

$$\frac{d\,G(0)}{dz} = \frac{d}{dz}\,H(h) + \frac{d}{dz}\,H(-h) = 0$$

and

$$\frac{d^2\,G(0)}{dz^2} = \frac{d^2}{dz^2}\,H_z(h) + \frac{d^2}{dz^2}\,H_z(-h) = 2\,\frac{d^2}{dz^2}\,H_z(h)$$

Thus the optimal solution is characterized as zero of the second derivative of $H_z(z)$.

$$\frac{d^2}{dz^2}\,H_x(z) = \frac{I\,R^2}{2}\,\frac{d^2}{dz^2}\,\left(\frac{1}{(R^2+z^2)^{3/2}}\right) = 0$$

We find

$$\begin{aligned} \frac{d}{dz}\,\frac{1}{(R^2+z^2)^{3/2}} &= \frac{-3\cdot 2\,z}{2\,(R^2+z^2)^{5/2}} = \frac{-3\,z}{(R^2+z^2)^{5/2}} \\ \frac{d^2}{dz^2}\,\frac{1}{(R^2+z^2)^{3/2}} &= \frac{-3\,(R^2+z^2)^{5/2} + 3\,z\,\frac{5}{2}(R^2+z^2)^{3/2}\,2\,z}{(R^2+z^2)^5} \\ &= \frac{-3\,(R^2+z^2) + 3\,z\,5\,z}{(R^2+z^2)^{7/2}} = 3\,\frac{4\,z^2-R^2}{(R^2+z^2)^{7/2}} \end{aligned}$$

Thus the second derivative of $G(0)$ vanishes if $h = \pm\frac{R}{2}$. This implies the the distance between the centers of the coil should be equal to the Radius $R$. This is confirmed by the results in Figure 5.

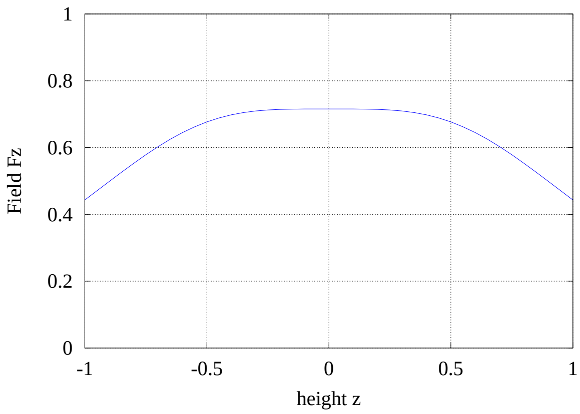


Figure 5: Magnetic field along the central axis, Helmholtz configuration

The above figures were generated by *Octave* with the help of a function file `HzAxis.m` to compute the field along the axis

**HzAxis.m**

```
function H = HzAxis(z)
  R = 1;
  H = R^2/2*(R^2+z.^2).^(-3/2);
endfunction
```

and the commands

**Octave**

```
z = linspace(-2,3,101);
figure(1);
dz=0.0;
plot(z,HzAxis(z-dz)+HzAxis(z+dz))
axis([-2,3,0,1])
grid on; xlabel('height z'); ylabel('Field Fz');

figure(2);
dz = 1;
plot(z,HzAxis(z-dz)+HzAxis(z+dz))
axis([-2,3,0,1])
grid on; xlabel('height z'); ylabel('Field Fz');
```

```
figure(3);
dz = 1/2;
plot(z,HzAxis(z-dz)+HzAxis(z+dz))
axis([-1,1,0,1])
grid on; xlabel('height z'); ylabel('Field Fz');
```

### 3.3 Field in the plane of the conductor

In the plane $z = 0$ of the conductor the field will show a radial symmetry again, the field at the point $(x, y, 0)$ is given by a rotation of the field at the point $(\sqrt{x^2 + y^2}, 0, 0)$. Thus we compute the field along the $x$–axis only. Based on equation (2) we have to compute the integrals

$$\vec{H}(x,y,z) = \frac{I}{4\pi} \int_0^{2\pi} \frac{1}{(R^2 - 2\,x\,R\,\cos\phi + x^2 + z^2)^{3/2}} \begin{pmatrix} z\,R\,\cos\phi \\ z\,R\,\sin\phi \\ R^2 - x\,R\,\cos\phi \end{pmatrix} d\phi$$

For the $z$ component $H_z$ we need to integrate a scalar valued function, depending on the variable angle $\phi$ and the parameters $R$, $x$, $y$ and $z$. We define an *Octave* function `dHz()`.

**dHz.m**

```
function res = dHz(phi,R,x,z)
  res = R*(R-x.*cos(phi))/sqrt(R^2-2*x.*R.*cos(phi)+x.^2 +z.^2).^3;
endfunction
```

Currently we are only interested in $_z$ along the $x$–axis, i.e. $y = z = 0$ and we want to compute

$$H_z(x,0,0) = \frac{I}{4\pi} \int_0^{2\pi} \frac{R^2 - x\,R\,\cos\phi}{(R^2 - 2\,x\,R\,\cos\phi + x^2)^{3/2}}\,d\phi$$

An analytical formula for this integral can be given, but the expression is very complicated. Thus we prefer a numerical approach. The integral to be computed will depend on the parameters $x$, $y$ and $R$ representing the position at which the magnetic field will be computed. We use function handles and anonymous functions to deal with these parameters for the integration. We examine a coil with diameter $R = 1$ and first compute the field for values $-0.5 \le x \le 0.8$ and then for $1.2 \le x \le 3$. The results are shown in Figure 6. Observe that the field is large if we are close to the wire at $x \approx R = 1$ and the $z$ component changes sign outside of the circular coil. As $x$ increases $H_z$ converges to 0 . This is confirmed by physical facts.

**Octave**

```
x  = -0.5:0.05:0.8;
Fz = zeros(size(x));
for k = 1:length(x)
  fz = @(al)dHz(al,1,x(k),0);      % define the anonymous function
  Fz(k) = quad(fz,0,2*pi)/(4*pi); % integrate
endfor

figure(1)
plot(x,Fz)
grid on
axis([-0.5 0.8 0 1.2]);
xlabel('position x'); ylabel('Field Fz');

x2 = 1.2:0.05:3;
Fz2 = zeros(size(x2));
for k = 1:length(x2)
  fz = @(al)dHz(al,1,x2(k),0);      % define the anonymous function
  Fz2(k) = quad(fz,0,2*pi)/(4*pi); % integrate
endfor

figure(2)
plot(x,Fz,x2,Fz2)
```

```
grid on
axis([-0.5 3 -0.6 1.2]);
xlabel('position x'); ylabel('Field Fz');
```

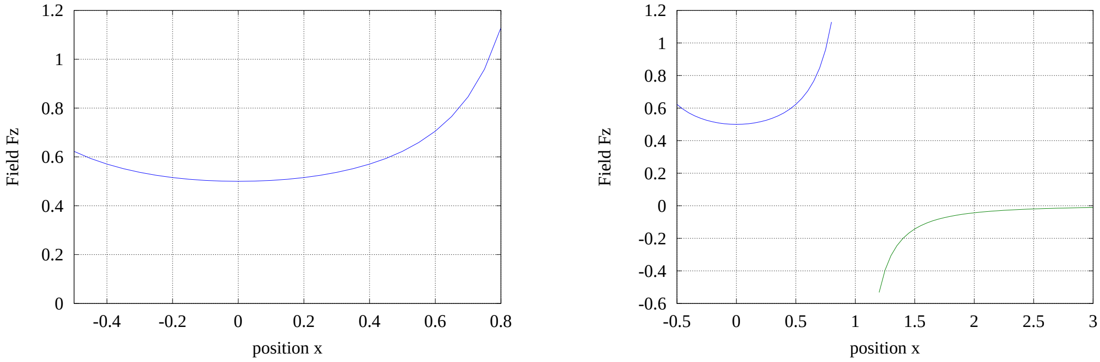


Figure 6: Magnetic field along the $x$ axis

### 3.4 Field in the $xz$–plane

In the $xz$ plane we have to examine both components of the field and thus we need to compute $H_x(x,0,y)$ too. Using equation (2) we find

$$H_x(x,0,z) = \frac{I}{4\,\pi} \int_0^{2\pi} \frac{z\,R\,\cos\,\phi}{(R^2 - 2\,x\,R\,\cos\,\phi + x^2 + z^2)^{3/2}}\,d\phi$$

We write a *Octave* function for the expression to be integrated.

**dHx.m**

```
function res = dHx(phi,R,x,z)
  res = R*z.*cos(phi)/sqrt(R^2-2*x.*R.*cos(phi)+x.^2 +z.^2).^3;
endfunction
```

We proceed as follows:

1. Choose the domain of $-0.4 \leq x \leq 0.7$ and $-1 \leq z \leq 2$. Generate vectors with the values of $x$ and $z$ for which the field $\vec{H}$ will be computed.
2. Generate a mesh of points with the command `meshgrid()` .
3. Create the empty matrices `Hx` and `Hz` for the components of the field.
4. Use a for loop to fill in the values of both components of $\vec{H}$, using the integrals based on (2).

**Octave**

```
z=-1:0.2:2; x=-0.4:0.1:0.7;
[xx,zz]=meshgrid(x,z);
x=xx(:); z=zz(:);
Hx=zeros(size(x)); Hz=Hx;

for k=1:length(x)
  fx=@(al)dHx(al,1,x(k),z(k));    % define the anonymous function
  Hx(k)=quad(fx,0,2*pi)/(4*pi);   % integrate
  fz=@(al)dHz(al,1,x(k),z(k));    % define the anonymous function
  Hz(k)=quad(fz,0,2*pi)/(4*pi);   % integrate
endfor
```

The next step is to visualize the vector field $\vec{H}$ in the $xz$ plane in the center of the circular coil. To arrive at Figure 7 we use the command `quiver()` to display the vector field.

In the left part of Figure 7 we find magnetic fields of drastically different sizes, in particular close to the conductor. For a good visualization it is often useful to normalize all vectors to have equal length. Find the result of the code below in the right part of Figure 7.

**Octave**

```
subplot(1,2,1)
quiver(x,z,Hx,Hz,0.2)
grid on
axis('equal')

scal = 1./sqrt(Hx(:).^2+Hz(:).^2);   % length of each vector
Hx = scal.*Hx; Hz = scal.*Hz;         % normalize length

subplot(1,2,2)
quiver(x,z,Hx,Hz,0.1)
grid on
axis('equal')
```

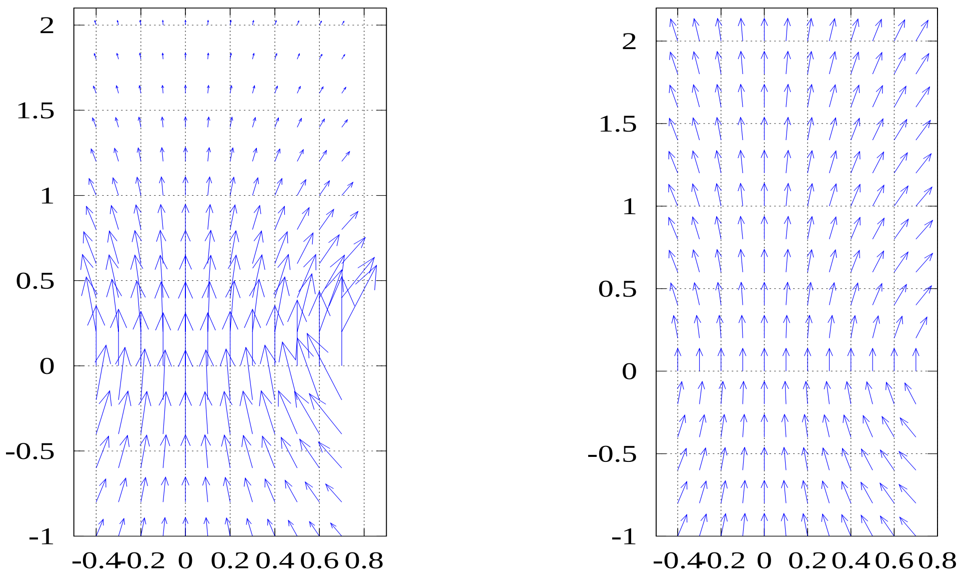


Figure 7: Magnetic vector field in a plane, actual length and normalized

# 4 The Helmholtz configuration

To obtain a homogeneous field in the center of two coils we have to place them at a distance equal to the radii of the circular coils. We can compute and visualize the field of this configuration using the same ideas and codes as in the section above. Find the result in Figure 8.

**Octave**

```
clf
x = -0.3:0.05:0.3; z = x;
h = 0.5; % optimal distance for Helmholtz configuration is 2*h=R

[xx,zz] = meshgrid(x,z);
x = xx(:); z = zz(:);
Hx = zeros(size(x)); Hz = Hx;
```

```
for k = 1:length(x)
  fx = @(al)(dHx(al,1,x(k),z(k)-h)+dHx(al,1,x(k),z(k)+h));
  Hx(k) = quad(fx,0,2*pi)/(4*pi);
  fz = @(al)(dHz(al,1,x(k),z(k)-h)+dHz(al,1,x(k),z(k)+h));
  Hz(k) = quad(fz,0,2*pi)/(4*pi);
endfor

quiver(x,z,Hx,Hz,0.001)
grid on
```

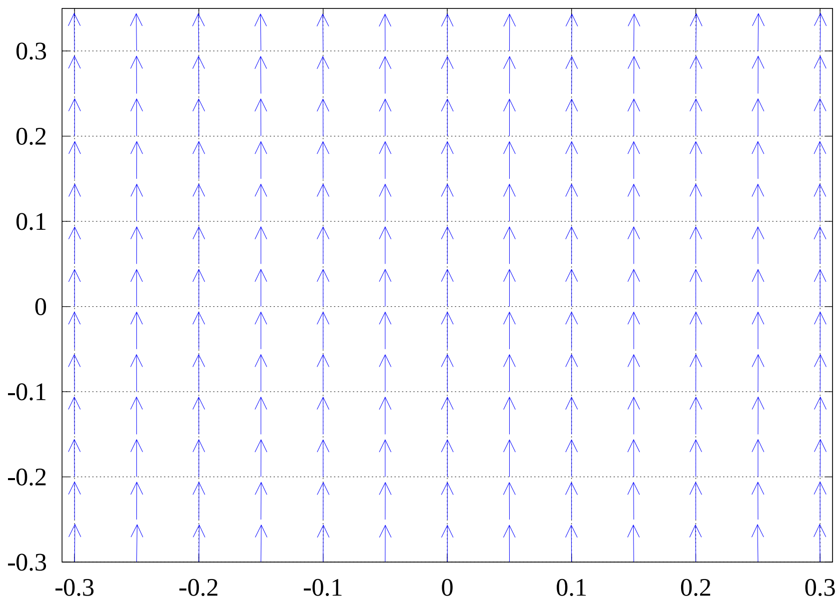


Figure 8: Magnetic field for two Helmholtz coils

## 4.1 Analysis of homogeneity of the magnetic field

The main purpose of the Helmholtz configuration is to provide a homogeneous field at the center of the coils. Thus we want the all components of the field to be constant. It is not obvious how to quantify the deviation from a constant magnetic field. We suggest three options:

1. Variation in $z$ component only
   The $z$ component $H_z$ is clearly the largest component. Thus we might examine the variations of $H_z$ only. One possible method is to generate level curves for the relative deviation

$$\text{relative deviation in } H_z \quad = \left|\frac{H_z(x,y,z) - H_z(0,0,0)}{H_z(0,0,0)}\right|$$

2. Variation in all components
   We might also take the other components into account and examine the deviation vector

$$\vec{H}(x,y,z) - \vec{H}(0,0,0) = \begin{pmatrix} H_x(x,y,z) \\ H_y(x,y,z) \\ H_z(x,y,z) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ H_z(0,0,0) \end{pmatrix}$$

   and then generate level curves for the relative deviation

$$\text{relative deviation in } \vec{H} \quad = \frac{\|\vec{H}(x,y,z) - \vec{H}_z(0,0,0)\|}{H_z(0,0,0)}$$

3. Variation in the strength of the magnetic field
   If only the strength $\|\vec{H}\|$ matters and the direction of the magnetic field is irrelevant we might examine level curves for

$$\text{relative deviation in strength } \|\vec{H}\| \quad = \left|\frac{\|\vec{H}(x,y,z)\| - H_z(0,0,0)}{H_z(0,0,0)}\right|$$

The above ideas can be implemented in *Octave*, leading to the result in Figure 9 for the relative deviation in $H_z$. The deviation is examined in the $xz$–plane, i.e. for $y = 0$. The result in Figure 9 for the relative deviation shows that the relative deviation is rather small at the center $(x,z) \approx (0,0)$ but increases shaply at the corners of the examined domain of $-0.3 \leq x \leq 0.3$ and $-0.3 \leq z \leq 0.3$.

**Octave**

```
n = sqrt(length(Hz))
HzM = reshape(Hz,n,n);
Hz0 = HzM(floor(n/2)+1,floor(n/2)+1)
reldev = abs(HzM-Hz0)/Hz0;
mesh(xx,zz,reldev)
```

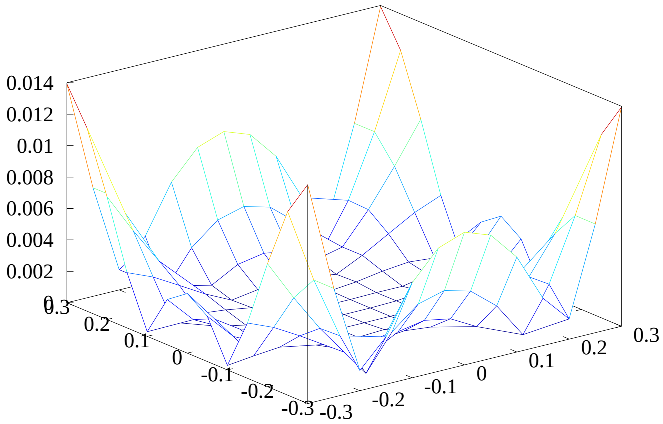


Figure 9: Relative changes of $H_z$ in the plane $y = 0$

## 4.2 Level curves for relative deviations

To estimate the domain for which the relative deviation is small we can use level curves. We use a fine mesh[1] to obtain smoother curves. In Figure 10 we find the level curves for relative deviation in $H_z$ at levels 0.001, 0.002, 0.005 and 0.01 . We see that the deviation remains small along a few lines, even if we move away from the center. This is confirmed by Figure 9. On might have to pay attention to the computation time. If a grid on $n \times n$ points is examined, then $2\,n^2$ integrals have to be computed. For the sample code below this translates to 5202 integrals.

**Octave**

```
n=51; x=linspace(-0.3,0.3,n); z=x;
h=0.5; % optimal distance for Helmholtz configuration

[xx,zz]=meshgrid(x,z);
x=xx(:); z=zz(:);
Hx=zeros(size(x)); Hz=Hx;
```

[1]The price to pay is some CPU time. For a quick check one may work at a lower resolution.

```
for k=1:length(x)
  fx=@(al)(dHx(al,1,x(k),z(k)-h)+dHx(al,1,x(k),z(k)+h));
  Hx(k)=quad(fx,0,2*pi)/(4*pi);
  fz=@(al)(dHz(al,1,x(k),z(k)-h)+dHz(al,1,x(k),z(k)+h));
  Hz(k)=quad(fz,0,2*pi)/(4*pi);
endfor
HzM = reshape(Hz,n,n);
Hz0 = HzM(floor(n/2)+1,floor(n/2)+1)
reldev = abs(HzM-Hz0)/Hz0;
contour(xx,zz,reldev,[0.001,0.002,0.005,0.01])
axis('equal'); grid on
xlabel('x'); ylabel('z');
```

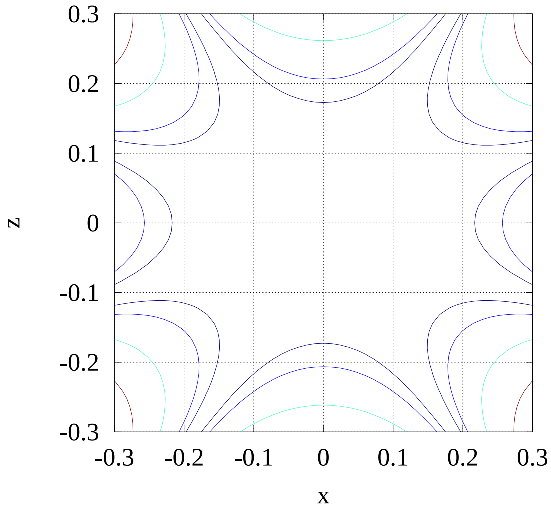


Figure 10: Level curves for the relative changes of $H_z$ at levels 0.001, 0.002, 0.005 and 0.01

It might be a better approach to examine the relative deviation in $\vec{H}$ and not the $z$ component only. Find the code below and the result in Figure 11 we see that the relative deviation of $\vec{H}$ increases uniformly as we move away from the center. For Helmholtz coils with radius $R = 1$ we find that in an approximate cylinder with radius 0.2 and height 0.4 the relative deviation in the magnetic field is smaller than 0.001, thus the field is rather homogeneous.

**Octave**

```
HxM = reshape(Hx,n,n);
HDev = sqrt(HxM.^2 + (HzM-Hz0).^2)/Hz0;
contour(xx,zz,HDev,[0.001,0.002,0.005,0.01])
axis('equal');grid on
xlabel('x'); ylabel('z');
```

# 5 Codes

In the previous sections the codes in Table 1 were used.

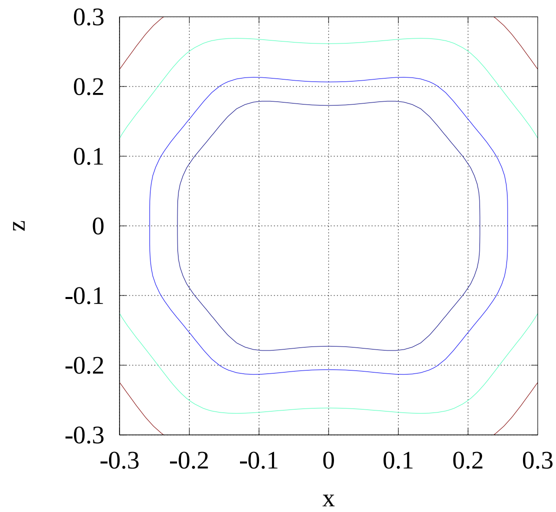


Figure 11: Level curves for the relative deviation of $\vec{H}$ at levels 0.001, 0.002, 0.005 and 0.01

| filename | function |
|---|---|
| `dHx.m`<br>`dHz.m` | function files for integrands of the $x$ and $y$<br>components of the magnetic field |
| `Hz_along_x_axis.m` | script file to compute $z$–component of field along $z$–axis |
| `VectorFields.m` | script file to compute vector field in $xz$–plane |
| `VectorFieldsHelmholtz.m`<br>`HelmholtzContours.m` | script file to compute vector field for Helmholtz configuration<br>script file to compute level curves for Helmholtz configuration |

Table 1: Codes for Biot-Savart and Helmholtz computations